38th Summer Conference on Topology and its Applications

Shift operators on Banach spaces

Maria Carvalho

Centre and Department of Mathematics

Faculty of Sciences of the University of Porto

July 22, 2024

This talk is based on a recent joint work with Udayan Darji (University of Louisville, USA) and Paulo Varandas (CMUP and Federal University of Bahia, Brazil), where we generalize the class of weighted shifts and the well-known dynamical information about them.

New operators

We consider linear operators defined by the following procedure:

- Take a Banach space X over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

- Consider a sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ of linear bounded invertible operators on X.

- Given a Banach space $\mathbf{B} \subset X^{\mathbb{Z}}$, and a vector in $\mathbf{B}$, apply the matrices $(S_n)_n$ coordinate wise and then shift the resulting sequence.

This way, we define a linear operator

$$\sigma_{\mathcal{S}} \colon \mathbf{B} \to \mathbf{B}$$

by

$$\sigma_{\mathcal{S}}\big((x_n)_{n\in\mathbb{Z}}\big) = \big(S_{n+1}(x_{n+1})\big)_{n\in\mathbb{Z}}.$$

We call such a map a shift operator generated by the sequence $\mathcal{S}$.

To ensure that $\sigma_{\mathcal{S}}$ is well defined and a bounded invertible operator we will always assume that there exists $C > 0$ such that

$$\max_{n \in \mathbb{Z}} \left\{ \|S_n\|, \, \|S_n^{-1}\| \right\} \, < \, C.$$

Example

Consider $d \in \mathbb{N}$, $X = \mathbb{R}^d$ endowed with the Euclidean norm and a sequence $(S_n)_{n \in \mathbb{Z}}$ of d-dimensional invertible matrices

$$S_n \colon \mathbb{R}^d \to \mathbb{R}^d.$$

Then, for every $1 \leq p < +\infty$, the shift operator $\sigma_{\mathcal{S}}$ may be defined on $\mathbf{B} = \ell_p(\mathbb{R}^d)$.

In particular, when $d = 1$ and $S_n \colon \mathbb{R} \to \mathbb{R}$ is given by

$$S_n(x) = \omega_n \, x$$

for a bounded sequence $\omega = (\omega_n)_{n \in \mathbb{Z}}$ of real numbers such that $\inf_{n \in \mathbb{Z}} |\omega_n| > 0$, then $\sigma_{\mathcal{S}}$ is precisely the bilateral weighted backward shift

$$B_\omega\big((x_n)_{n \in \mathbb{Z}}\big) = \big(\omega_{n+1} \, x_{n+1}\big)_{n \in \mathbb{Z}}.$$

Inspiring setting

The family of weighted shifts has been playing an essential role in the study of linear dynamics, resembling the importance symbolic dynamics attained in topological dynamics and ergodic theory.

It was a simple matter of choosing proper weights to be able to distinguish canonical dynamical properties such as transitivity, weak mixing, mixing and chaoticity by using weighted shifts.

Moreover, weighted shifts helped to determine whether well-known results or dynamical properties of classical linear dynamics in finite dimension hold in the setting of infinite dimensional Banach spaces.

Hyperbolicity

For instance, in finite dimensional Banach spaces, hyperbolicity[1] is a Baire generic property in the set of linear bounded invertible operators.

Besides, in this finite dimensional setting,[2]

hyperbolicity $\Leftrightarrow$ shadowing property $\Leftrightarrow$ structural stability.

[1] A linear bounded invertible operator $T\colon X \to X$ is hyperbolic if there are closed $T-$invariant subspaces M and N of X such that $X = M \oplus N$ and $T\mid_M$, $T^{-1}\mid_N$ are uniform contractions.

[2] J. Ombach. *The shadowing lemma in the linear case.* 1994

Shadowing

What is the relationship between hyperbolicity and the shadowing property[3] in infinite dimensional Banach spaces?

It is known that we always have

$$\text{hyperbolicity} \quad \Rightarrow \quad \text{shadowing property.}$$

Yet, until 2018, it was an open problem whether the shadowing property implies hyperbolicity in infinite dimensional spaces.

[3] T has the shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any sequence $(x_n)_{n\in\mathbb{Z}}$ satisfying $\|T(x_n) - x_{n+1}\| \leq \delta$ for all $n \in \mathbb{Z}$ there exists $x \in X$ such that $\|T^n(x) - x_n\| \leq \varepsilon$ for all $n \in \mathbb{Z}$.

This was settled negatively[4] by constructing a weighted backward shift which has the shadowing property, is structurally stable, but is not hyperbolic.

[4] N. Bernardes *et al. Expansivity and shadowing in linear dynamics.* 2018

In the process of establishing this result, a weaker[5] notion of hyperbolicity arose, nowadays called generalized hyperbolicity.

It turns out that[6]

generalized hyperbolicity $\Rightarrow$ shadowing property.

[5]A linear bounded invertible operator $T\colon X \to X$ is generalized hyperbolic if there are closed subspaces M and N of X such that $T(M) \subset M$, $T^{-1}(N) \subset N$, $X = M \oplus N$ and $T|_M$, $T^{-1}|_N$ are uniform contractions.

[6]P. Cirilo *et al.* *Dynamics of generalized hyperbolic linear operators.* 2021

A characterization of weighted backward shifts on $\ell^p(\mathbb{Z})$, $1 \leq p < +\infty$, or $c_0(\mathbb{Z})$ with the shadowing property was given by Bernardes and Messaoudi[7], yielding that, for weighted backward shifts,

$$\text{generalized hyperbolicity} \quad \Leftrightarrow \quad \text{shadowing property.}$$

It was also established that we always have

$$\text{generalized hyperbolicity} \quad \Rightarrow \quad \text{structural stability}$$

and

$$\text{hyperbolicity} \quad \Leftrightarrow \quad \text{shadowing property} + \text{expansivity.}$$

[7] N. Bernardes *et al. Shadowing and structural stability for operators.* 2021; N. Bernardes *et al. A generalized Grobman-Hartman theorem.* 2022

Koopman operators

Apart from the class of weighted backward shifts, the equivalence between generalized hyperbolicity and the shadowing property was also established for a large family of operators in L^p spaces, the well-studied composition operators[8].

[8] E. D'Aniello *et al. Generalized hyperbolicity and shadowing in L^p spaces.* 2021

Shift-like operators

Moreover, for a special type of them, called shift-like operators[9], it was shown that such operators admit a weighted backward shift as a factor, and that the operator satisfies a given property $\mathcal{P}$ if and only if its weighted backward shift factor does, where $\mathcal{P}$ is one of numerous properties, such as shadowing, expansivity, transitivity, mixing, etc.

[9] E.D'Aniello *et al. Shift-like operators on* $\ell^p(X)$. 2022

Open question

So far, all known examples of linear invertible bounded operators on Banach spaces with the shadowing property are generalized hyperbolic.

This motivates the question whether, conversely, every linear invertible bounded operator on a Banach space satisfying the shadowing property is generalized hyperbolic.

We expect that the extension of weighted backward shifts I am reporting on may pave the way for relevant new information regarding this and other questions.

New results: a rich family of examples

We have proved that the class of shift operators comprises, up to linear conjugation:

- linear bounded invertible dissipative[10] operators,
- shift-like operators,
- finite products of weighted backward shifts,
- some skew-products of weighted backward shifts.

[10] $T : X \to X$ is dissipative if there exists a closed subspace $E_0 \subset X$ such that the collection $\left(T^n(E_0)\right)_{n \in \mathbb{Z}}$ satisfies $X = \overline{\text{span}\left(\oplus_{n \in \mathbb{Z}} E_n\right)}$, where $E_n = T(E_0)$.

New results: a classification

We have classified large classes of shift operators, including those generated by orthogonal matrices, diagonalizable matrices, hyperbolic matrices and suitable combinations of these.

This classification yields verifiable conditions which we used to construct examples of shift operators with a variety of dynamical properties.

As a consequence, we have shown that, for relevant new classes of linear operators,

$$\text{generalized hyperbolicity} \quad \Leftrightarrow \quad \text{shadowing property.}$$

1. Dissipative operators

Our first result shows that the family of shift operators includes, up to linear conjugation, the dissipative ones.

Theorem 1 (MC, U. Darji, P. Varandas (2024))

Suppose that $T\colon X \to X$ is a linear bounded invertible dissipative operator on a Banach space X. Then T is linearly conjugate to a shift operator $\sigma_{\mathcal{S}}\colon \mathbf{B} \to \mathbf{B}$, where $\mathbf{B}$ is a Banach space, $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$ and S_n is the identity map for every $n \in \mathbb{Z}$.

Remark 1

The proof of Theorem 1 does not provide, in general, an explicit description of the Banach space **B** where the shift operator is defined.

However, under additional assumptions or within specific frameworks, we succeeded in describing this underlying space.

One such framework is the class of composition operators.

Remark 2

Even though all the operators S_n are the identity map, the shift operator $\sigma_{\mathcal{S}}$ can exhibit a variety of dynamical behaviors.

For instance, as happens with subshifts in symbolic dynamics, the shift operator $\sigma_{\mathcal{S}}$ may or may not have the shadowing property, be or not be transitive, be or not be expansive, etc., depending on the underlying space.

2. Main question

May we know in advance, from properties of the sequence $(S_n)_n$, if the corresponding shift operator has non-trivial recurrence, or has the shadowing property, or is hyperbolic, or generalized hyperbolic, or expansive, etc.?

Orthogonal basis

Let us see our answer to this question for a special class of sequences $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$.

Definition 2

Suppose that X is a Hilbert space. We say that a sequence $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$ of linear bounded invertible operators on X has an orthogonal basis $\mathcal{E}$ in X if $\{e_n(b)\colon\ b\in\mathcal{E}\}$ is orthogonal for every $n\in\mathbb{Z}$, where, for each $b\in\mathcal{E}$,

$$e_n(b) = \begin{cases} \dfrac{S_n^{-1}\circ\ldots\circ S_1^{-1}(b)}{\| S_n^{-1}\circ\ldots\circ S_1^{-1}(b)\|} & \text{if } n\geq 1 \\ \dfrac{b}{\|b\|} & \text{if } n=0 \\ \dfrac{S_{n+1}\circ\ldots\circ S_0(b)}{\| S_{n+1}\circ\ldots\circ S_0(b)\|} & \text{if } n\leq -1 \end{cases}.$$

Moving referential

We note that, by construction, if $S_n = L$ for all $n \in \mathbb{Z}$, then $(e_n(b))_{n\in\mathbb{Z}}$ is the normalized orbit of $b \in \mathcal{E}$ by L.

Moreover, in general one has

$$S_{n+1}\left(e_{n+1}(b)\right) = \omega_{n+1}(b)\, e_n(b) \qquad \forall\, n \in Z, \quad \forall\, b \in \mathcal{E}$$

where

given $x \in X \setminus \{0\}$, the sequence of weights

$$\omega(x) = (\omega_n(x))_{n \in \mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$$

is defined by

$$\omega_n(x) = \begin{cases} \frac{\|x\|}{\|S_1^{-1}(x)\|} & \text{if } n = 1 \\ \frac{\|S_{n-1}^{-1} \dots S_1^{-1}(x)\|}{\|S_n^{-1} \dots S_1^{-1}(x)\|} & \text{if } n > 1 \\ \frac{\|S_0(x)\|}{\|x\|} & \text{if } n = 0 \\ \frac{\|S_n \dots S_0(x)\|}{\|S_{n+1} \dots S_0(x)\|} & \text{if } n < 0. \end{cases}$$

Suitable weights

Since we are assuming that there exists $C > 0$ such that

$$\max_{n \in \mathbb{Z}} \left\{ \|S_n\|, \, \|S_n^{-1}\| \right\} \, < \, C$$

we are sure that the sequence $\big(\omega_n(x)\big)_{n \in \mathbb{Z}}$ is bounded and satisfies

$$\inf_{n \in \mathbb{Z}} \, |\omega_n| \, > \, 0.$$

Example

If $d \in \mathbb{N}$ and S_n is an orthogonal $d \times d$ matrix in $\mathbb{R}^d$ for every $n \in \mathbb{Z}$, then any orthogonal basis $\mathcal{E}$ of X is an orthogonal basis for the sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$.

In this case, it is known that each S_n is a scalar multiple of an isometry, say $S_n = \alpha_n I_n$, with $\alpha_n \in \mathbb{R} \setminus \{0\}$ and I_n an isometry.

Thus, $\omega_n(x) = |\alpha_n|$ for every $n \in \mathbb{Z}$ and $x \in X \setminus \{0\}$.

3. Factors of shift operators

By the next result we find weighted backward shifts defined on $\ell_p(\mathbb{K})$, for $1 \leq p < +\infty$, as factors of shift operators.

Theorem 3 (MC, U. Darji, P. Varandas (2024))

Let X be a Hilbert space over the field $\mathbb{K}$ and suppose that the sequence of operators $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ on X has an orthogonal basis $\mathcal{E}$. Then, given $1 \leq p < +\infty$, for any $b \in \mathcal{E}$ the weighted backward shift

$$B_{\omega(b)} \colon \ell_p(\mathbb{K}) \to \ell_p(\mathbb{K})$$

is a factor of the shift operator $\sigma_{\mathcal{S}} \colon \ell_p(X) \to \ell_p(X)$.

Consequence

The following immediate consequence of the previous result is due to the fact that, in general, factors preserve transitivity, mixing and the shadowing property.

Corollary 4

Under the assumptions of the previous theorem, if $\sigma_{\mathcal{S}}$ is transitive, mixing or has the shadowing property, then the same holds for $B_{\omega(b)}$ for all $b \in \mathcal{E}$.

Accordingly, to prove that $\sigma_{\mathcal{S}}$ does not have a property in question (transitive, mixing or shadowing), we may simply find $b \in \mathcal{E}$ such that $B_{\omega(b)}$ does not have the corresponding property.

4. Finite products of weighted shifts

Given a finite number of copies of the Banach space $\ell_p(\mathbb{K})$, we endow the product vector space with the sum norm.

The next result provides sufficient conditions for a shift operator to be linearly conjugate to a finite product of weighted backward shifts.

Classification theorem I

Theorem 5 (MC, U. Darji, P. Varandas (2024))

Let X be a finite dimensional Hilbert space over the field $\mathbb{K}$ and assume that the sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ has an orthogonal basis $\mathcal{E}$. Then, for each $1 \leq p < +\infty$, the operator $\sigma_{\mathcal{S}} \colon \ell_p(X) \to \ell_p(X)$ is linearly conjugate to the finite product of weighted backward shifts

$$\prod_{b \in \mathcal{E}} B_{\omega(b)} \colon \left(\ell_p(\mathbb{K})\right)^{\dim X} \quad \to \quad \left(\ell_p(\mathbb{K})\right)^{\dim X}$$

where $\dim X$ stands for the dimension of X.

Remark 1

One may wonder why the weights ω_n are real numbers despite the fact that the underlying field $\mathbb{K}$ may be $\mathbb{C}$.

Actually, it is well-known that every weighted backward shift on $\mathbb{C}$ with complex weights is linearly conjugate to a weighted backward shift with positive weights.[11].

Moreover, every weighted backward shift with positive weights over $\mathbb{C}$ can easily be seen as linearly conjugate to the product of two weighted backward shifts with positive weights over $\mathbb{R}$.

[11] F. Martínez-Giménez, A. Peris. *Chaos for backward shift operators.* 2002

Remark 2

The assumptions on the previous theorem seem restrictive.

So we looked for alternative statements where orthogonality might be replaced by more general assumptions.

Given a Banach space X and a vector $x \in X$, let $F_x \subset X$ denote the subspace spanned by x.

Consider a basis $\mathcal{E}$ of X, $b \in \mathcal{E}$ and the projection $\Pi_b \colon X \to F_b$ given by

$$\Pi_b \left(\sum_{e \in \mathcal{E}} \alpha_e \, e \right) = \alpha_b \, b$$

for any sequence of scalars $(\alpha_e)_{e \in \mathcal{E}}$ in $\mathbb{K}$.

For every $b \in \mathcal{E}$, the map Π_b is well defined, since every vector in X has a unique representation in terms of vectors in $\mathcal{E}$; moreover, Π_b is linear, bounded and $\Pi_b \circ \Pi_b = \Pi_b$.

5. $\mathcal{S}-$bounded projections

Definition 6

Given a sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ of linear bounded invertible operators on a Banach space X, we say that a basis $\mathcal{E}$ of X has $\mathcal{S}-$bounded projections if there exists $C > 0$ such that

$$\sup_{n \in \mathbb{Z}} \sup_{b \in \mathcal{E}_n} \|\Pi_b\| \leq C < +\infty$$

where $\mathcal{E}_n = \{e_n(x) \colon x \in \mathcal{E}\}$.

Classification theorem II

Theorem 7 (MC, U. Darji, P. Varandas (2024))

Let X be a finite dimensional Banach space, $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ be a sequence of linear bounded invertible operators on X and $\mathcal{E}$ be a basis of X with $\mathcal{S}-$bounded projections. Then, for each $1 \leq p < +\infty$, the shift operator $\sigma_{\mathcal{S}} \colon \ell_p(X) \to \ell_p(X)$ is linearly conjugate to the finite product of weighted backward shifts

$$\prod_{b \in \mathcal{E}} B_{\omega(b)} \colon (\ell_p(\mathbb{K}))^{\dim X} \to (\ell_p(\mathbb{K}))^{\dim X}.$$

Consequence 1

The previous theorem allows us to strengthen the Classification theorem I by replacing orthogonality with suitable uniform lower and upper bounds on the angles between distinct vectors in each basis $\mathcal{E}_n$.

Corollary 8

Let X be a finite dimensional Hilbert space with dimension $\dim X \geq 2$ and assume that $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ has a basis $\mathcal{E}$ of X for which there exists

$$0 \,<\, \gamma \,<\, 1/(\dim X - 1)$$

such that, for every $n \in \mathbb{Z}$, the angle $\measuredangle(u, v)$ between distinct vectors $u, v \in \mathcal{E}_n$ satisfies the condition

$$\cos \measuredangle(u, v) \,\in\, [-\gamma, \gamma].$$

Then, $\mathcal{E}$ has $\mathcal{S}-$bounded projections and so, given $1 \leq p < +\infty$, the shift $\sigma_{\mathcal{S}} \colon \ell_p(X) \to \ell_p(X)$ is linearly conjugate to the finite product

$$\prod_{b \in \mathcal{E}} B_{\omega(b)} \colon \, \big(\ell_p(\mathbb{K})\big)^{\dim X} \,\to\, \big(\ell_p(\mathbb{K})\big)^{\dim X}.$$

Observe that, as d goes to infinity, the requirement in the previous corollary gets closer to orthogonality.

Consequence 2

The next corollary provides a way of improving the Classification theorem I by establishing another sufficient condition, which does not depend on the dimension d of the Hilbert space X, for a sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ to have $\mathcal{S}-$bounded projections.

Given a Hilbert space X, a subspace $F \subset X$ and a vector $v \in X \setminus \{0\}$, consider the infimum

$$\measuredangle(v, F) = \inf_{u \in F \setminus \{0\}} \measuredangle(v, u).$$

For a basis $\mathcal{E}$ of X and $v \in \mathcal{E}$, we denote by $F_{n,v}$ the subspace of X generated by the vectors

$$\{e_n(b) \colon b \in \mathcal{E} \setminus \{v\}\}.$$

Corollary 9

Let X be a finite dimensional Hilbert space with dimension $\dim X \geq 2$ *and assume that* $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ *has a basis* $\mathcal{E}$ *of* X *satisfying*

$$\inf_{n \in \mathbb{Z},\ v \in \mathcal{E}} \measuredangle(e_n(v), F_{n,v}) \ > \ 0.$$

Then, $\mathcal{E}$ *has* $\mathcal{S}-$*bounded projections and so, for every* $1 \leq p < +\infty$, *the shift* $\sigma_{\mathcal{S}} \colon \ell_p(X) \to \ell_p(X)$ *is linearly conjugate to the finite product*

$$\prod_{b \in \mathcal{E}} B_{\omega(b)} \colon \big(\ell_p(\mathbb{K})\big)^{\dim X} \ \to \ \big(\ell_p(\mathbb{K})\big)^{\dim X}.$$

Consequence 3

As previously mentioned, some dynamical properties of linear bounded invertible operators may be conveyed to factors. We may strengthen such results in the setting of Classification theorem II.

Let $\mathcal{Q}$ be one of these properties: mixing, shadowing or generalized hyperbolicity. Then:

Corollary 10

Let X be a finite dimensional Banach space, $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ be a sequence of linear bounded invertible operators on X and $\mathcal{E}$ be a basis of X with $\mathcal{S}-$bounded projections. Then the shift $\sigma_{\mathcal{S}} : \ell_p(X) \to \ell_p(X)$ has property $\mathcal{Q}$ if and only if $B_{\omega(b)}$ has property $\mathcal{Q}$ for all $b \in \mathcal{E}$.

Remark

It is easy to verify that the previous corollary fails for the shadowing property in case X is infinite dimensional.

For instance, let $X = \ell_2(\mathbb{R})$, $\mathcal{E} = \{e_j : j \in \mathbb{Z}\}$ be the canonical basis of X and $T : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be defined by

$$T(e_j) = \left(1 + \frac{1}{2^j}\right) e_j$$

for all $j \in \mathbb{Z}$.

Then the shift operator $\sigma_{\mathcal{S}} : \ell_p(X) \to \ell_p(X)$, determined by $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ with $S_n = T$ for every $n \in \mathbb{Z}$, does not have the shadowing property according to Bernardes and Messaoudi criteria, although for all $b \in \mathcal{E}$ the weighted shift $B_{\omega(b)} : \ell_p(\mathbb{R}) \to \ell_p(\mathbb{R})$ has it.

Consequence 4

Since generalized hyperbolic linear bounded invertible operators satisfy the shadowing property; the shadowing property is preserved by factors; a weighted backward shift on $\ell_p(\mathbb{K})$ has the shadowing property if and only if it is generalized hyperbolic; and a finite product of generalized hyperbolic operators is generalized hyperbolic, one has:

Corollary 11

Let X be a finite dimensional Banach space, $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$ be a sequence of linear bounded invertible operators on X and $\mathcal{E}$ be a basis of X with $\mathcal{S}-$bounded projections. Then the shift $\sigma_{\mathcal{S}} : \ell_p(X) \to \ell_p(X)$ is generalized hyperbolic if and only if it has the shadowing property.

Consequence 5

Applying Bernardes-Messaoudi characterization of the weighted backward shifts with the shadowing property to the setting of the previous corollary, and rewriting the ω_n's in terms of the S_n's, we have an immediate criteria for shadowing within the class of shift operators.

Example

A shift operator $\sigma_{\mathcal{S}}$ may fail to satisfy the shadowing property even though each of the operators S_n in the sequence $\mathcal{S}$ has that property, as the next example illustrates.

Consider $X = \mathbb{R}^2$ and the Banach space

$$\ell^\infty(\mathbb{R}^2) = \Big\{(x_n, y_n)_n \in \left(\mathbb{R}^2\right)^{\mathbb{Z}} \colon \sup_{n \in \mathbb{Z}} \|(x_n, y_n)\| < +\infty\Big\}.$$

Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear invertible operator given by

$$(x, y) \in \mathbb{R}^2 \qquad \mapsto \qquad T(x, y) = \left(2x, \frac{1}{2}y\right).$$

The map T is hyperbolic, hence satisfies the shadowing property.

Consider now the shift operator $\sigma_{\mathcal{S}}$ where $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ is defined by

$$S_n = \begin{cases} T & \text{if } n \text{ is odd} \\ T^{-1} & \text{otherwise.} \end{cases}$$

Then this shift operator does not have the shadowing property in view of the aforementioned criteria.

6. Applications

To illustrate the scope of our results, let us discuss a few examples.

Example 1

Consider a rotation matrix in $X = \mathbb{R}^2$ given by

$$R_\theta = \begin{pmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix}$$

where $\theta \in \,]0,1[$.

Let $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$ where $S_n = \left(1/2\right) R_{\theta_n}$ for all $n \in \mathbb{Z}$ and $\theta_n \in \,]0,1[$.

Then any orthogonal basis of $\mathbb{R}^2$ is an orthogonal basis for $\mathcal{S}$.

Therefore $\sigma_{\mathcal{S}} \colon \ell_p(\mathbb{R}^2) \to \ell_p(\mathbb{R}^2)$ is linearly conjugate to the product of weighted backward shifts $B_\omega \times B_\omega$, where $\omega = (\omega_n)_{n \in \mathbb{Z}}$ is given by

$$\omega_n = \frac{1}{2} \qquad \forall\, n \in \mathbb{Z}.$$

Moreover, $\sigma_{\mathcal{S}}$ is hyperbolic (so it has the shadowing property and is expansive).

Example 2

Take $d \in \mathbb{N}$, $X = \mathbb{R}^d$ and invertible diagonal matrices

$$S_n = \begin{pmatrix} \lambda_n(1) & 0 & \dots & 0 \\ 0 & \lambda_n(2) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \lambda_n(d) \end{pmatrix}.$$

Clearly, the canonical basis of $\mathbb{R}^d$ is an orthogonal basis for the sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$. So the corresponding shift operator $\sigma_{\mathcal{S}} \colon \ell_p(\mathbb{R}^d) \to \ell_p(\mathbb{R}^d)$ is linearly conjugate to a finite product of weighted backward shifts.

Moreover, $\sigma_{\mathcal{S}}$ has the shadowing property if and only if $(\lambda_n(t))_{n\in\mathbb{Z}}$ satisfies Bernardes-Messaoudi criteria, for every $1 \leq t \leq d$.

Similarly, $\sigma_{\mathcal{S}}$ has the mixing property if and only if $(\lambda_n(t))_{n\in\mathbb{Z}}$ satisfies Grosse-Erdmann-Maguillot criteria.[12]

[12] K.-G Grosse-Erdmann, A. Peris Manguillot. Linear Chaos. Universitext, Springer, 2011

We note that, conversely, a product of d weighted backward shifts, each defined in $\ell_p(\mathbb{K})$, $1 \leq p < +\infty$, is linearly conjugate to the shift operator $\sigma_{\mathcal{S}}$ generated by the sequence $(S_n)_{n \in \mathbb{Z}}$ of diagonal matrices with respect to the canonical basis in $\mathbb{K}^d$, whose entries are precisely the weights of the factors.

Example 3

Let $X = \mathbb{R}^2$ and consider

$$L = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The matrix L is not orthogonal, but is diagonalizable with eigenvalues

$$\frac{3 \pm \sqrt{5}}{2}$$

and eigenvectors

$$\mathcal{E} = \left\{ \left(1, \frac{\sqrt{5}-1}{2}\right), \left(1, -\frac{\sqrt{5}+1}{2}\right) \right\}$$

which are orthogonal and define $L-$invariant directions.

Therefore, the shift $\sigma_{\mathcal{S}} \colon \ell_p(\mathbb{R}^2) \to \ell_p(\mathbb{R}^2)$ has an orthogonal basis, so it is linearly conjugate to the product of the weighted backward shifts $B_\omega \times B_{\bar{\omega}}$, where $\omega = (\omega_n)_{n \in \mathbb{Z}}$, $\bar{\omega} = (\bar{\omega}_n)_{n \in \mathbb{Z}}$,

$$\omega_n = \frac{3+\sqrt{5}}{2} \qquad \text{and} \qquad \bar{\omega}_n = \frac{3-\sqrt{5}}{2} \qquad \forall\, n \in \mathbb{Z}.$$

Thus, $\sigma_{\mathcal{S}}$ is hyperbolic.

Operators without an orthogonal basis

Definition 12

Let X be a finite dimensional Banach space. A sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ of operators on X is said to be jointly diagonalizable if there is a linear invertible operator $\mathcal{L}$ such that, for every $n \in \mathbb{Z}$, there exists a diagonal operator D_n satisfying

$$S_n = \mathcal{L} \, D_n \, \mathcal{L}^{-1}.$$

From the information regarding sequences of diagonal matrices, one deduces that:

Proposition 1

Let X be a Banach space with dimension $1 \leq d < +\infty$. Suppose that the sequence $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ of operators on X is jointly diagonalizable. Then, for $1 \leq p < +\infty$, the shift $\sigma_{\mathcal{S}} \colon \ell_p(X) \to \ell_p(X)$ is linearly conjugate to the product of d weighted backward shifts, each one defined on $\ell_p(\mathbb{K})$.

Example 4

Let $X = \mathbb{R}^2$ and consider the sequence $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$ of operators in X such that $S_n = L$ for every $n \in \mathbb{Z}$, where

$$L = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

The eigenvalues of L are $\lambda = 2 \pm \sqrt{3}$ with eigenvectors $(\pm\sqrt{3}, 1)$, respectively.

As the eigenvectors are not orthogonal and the matrix L is hyperbolic, the image by L of any two orthogonal vectors is not orthogonal. Consequently, $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$ does not have an orthogonal basis.

However, as $(S_n)_{n\in\mathbb{Z}}$ is jointly diagonalizable, we can still conclude that $\sigma_{\mathcal{S}}$ is linearly conjugate to the product of two weighted backward shifts.

Moreover, $\sigma_{\mathcal{S}}$ is hyperbolic.

Example 5

Let $X = \mathbb{R}^2$ and let us go back to the hyperbolic matrix

$$L = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let $0 < \lambda_- < 1 < \lambda_+$ be the eigenvalues of L with eigenvectors v^- and v^+, respectively, and $\mathbb{R}^2 = E^+ \oplus E^-$ be the orthogonal $L-$invariant splitting given by the eigenspaces E^+ and E^- of L.

We may choose sequences of bounded matrices $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ inside a small open neighborhood $\mathcal{U}$ of L in such a way that

- $S_0 = Id, \qquad S_{-n} \in \mathcal{U} \quad \text{and} \quad S_n \in \mathcal{U} \qquad \forall\, n \in \mathbb{N};$

and

- there is a basis $\mathcal{E}$ which has $\mathcal{S}-$bounded projections.

Therefore, the shift operator $\sigma_{\mathcal{S}}$ is linearly conjugate to the product of two weighted backward shifts, which are hyperbolic.

In particular, $\sigma_{\mathcal{S}}$ has the shadowing property.

Example 6

Combining hyperbolic and elliptic matrices we may construct examples of shift operators linearly conjugate to a finite product of weighted backward shifts, though some of these examples satisfy the shadowing property while others do not.

For instance, consider the linear operators on $\mathbb{R}^2$ given by the matrices

$$R_{2\pi\zeta} = \begin{pmatrix} \cos(2\pi\zeta) & -\sin(2\pi\zeta) \\ \sin(2\pi\zeta) & \cos(2\pi\zeta) \end{pmatrix}, \quad \zeta \in \mathbb{R} \setminus \mathbb{Q}$$

and

$$L = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

We may define sequences $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ of operators on $\mathbb{R}^2$ where $S_0 = Id$ and, for each $n \in \mathbb{Z} \setminus \{0\}$, S_n is either $R_{2\pi\zeta}$ or L,

$$\cdots \underbrace{L \dots L}_{n_2} \; \underbrace{R_{2\pi\zeta} \dots R_{2\pi\zeta}}_{m_1} \; \underbrace{L \dots L}_{n_1} \; \overset{\downarrow}{Id} \; \underbrace{L \dots L}_{n_1} \; \underbrace{R_{2\pi\zeta} \dots R_{2\pi\zeta}}_{m_1} \; \underbrace{L \dots L}_{n_2} \cdots$$

for suitable choices of sequences of positive integers $(n_i)_{i \in \mathbb{N}}$ and $(m_i)_{i \in \mathbb{N}}$, where $\downarrow$ marks the position zero, to ensure that there is a basis $\mathcal{E}$ which has $\mathcal{S}-$bounded projections.

So the shift operator $\sigma_{\mathcal{S}}$ is linearly conjugate to the product of two weighted backward shifts.

We may now change the frequency with which L occurs in the sequence $(S_n)_{n\in\mathbb{Z}}$ to study its impact on the properties of the shift operator $\sigma_{\mathcal{S}}$.

We have proved that:

- Unbounded gaps: $\sup_{i\in\mathbb{N}} m_i = +\infty$.

In this case, $\sigma_{\mathcal{S}}$ does not satisfy the shadowing property.

- Bounded gaps: $\sup_{i\in\mathbb{N}} m_i < +\infty$.

In this case, $\sigma_{\mathcal{S}}$ has the shadowing property.

Example 7

Let $X = \mathbb{R}^2$ endowed with the norm given by $\|(x, y)\| = \max\{|x|, |y|\}$ and $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ be the sequence

$$S_n = L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \forall\, n \in \mathbb{Z}.$$

Then $\sigma_{\mathcal{S}} \colon \ell_p(\mathbb{R}^2) \to \ell_p(\mathbb{R}^2)$ is given by

$$\sigma_{\mathcal{S}}\big((x_n, y_n)_{n \in \mathbb{Z}}\big) = \big(x_{n+1} + y_{n+1},\, y_{n+1}\big)_{n \in \mathbb{Z}}.$$

More generally, for every $(x_n, y_n)_{n \in \mathbb{Z}} \in \ell_p(\mathbb{R}^2)$ and $k \in \mathbb{Z}$,

$$\sigma_{\mathcal{S}}^k\big((x_n, y_n)_{n \in \mathbb{Z}}\big) \;=\; \big(x_{n+k} + k\, y_{n+k},\, y_{n+k}\big)_{n \in \mathbb{Z}}.$$

In particular, we deduce that $\sigma_{\mathcal{S}}$ has unbounded orbits, such as the orbit of the vector $v = (0, y_n)_{n \in \mathbb{Z}} \in \ell_p(\mathbb{R}^2)$ with $y_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $y_0 = 1$, for which one has

$$\lim_{k \to +\infty} \|\sigma_{\mathcal{S}}^k(v)\|_p \;=\; +\infty.$$

Observe that, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
e_{-n}(1,0) &= \frac{L^n(1,0)}{\|L^n(1,0)\|} = (1,0) \\
e_{-n}(0,1) &= \frac{L^n(0,1)}{\|L^n(0,1)\|} = \frac{(n,1)}{n} = \Big(1,\ \frac{1}{n}\Big).
\end{aligned}
$$

Hence, the angle between these two vectors goes to zero as $n \to +\infty$ and

$$
\begin{aligned}
\big\|\Pi_{e_{-n}(0,1)}\big\| &\geq \frac{\|\Pi_{e_{-n}(0,1)}\big((0,1)\big)\|}{\|(0,1)\|} \\
&= \big\|\Pi_{e_{-n}(0,1)}\big(-n\, e_{-n}(1,0) + n\, e_{-n}(0,1)\big)\big\| \\
&= n.
\end{aligned}
$$

Consequently, the canonical basis $\mathcal{E} = \{(1,0),(0,1)\}$ does not have $\mathcal{S}-$bounded projections.

In this case,

$$\omega_n\big((1,0)\big) \;=\; 1 \qquad \forall\, n \in \mathbb{Z}$$

and

$$\omega_n\big((0,1)\big) \;=\; \begin{cases} 1 & \text{if } n = 0, 1 \\[2em] \frac{n-1}{n} & \text{if } n \geq 2 \\[2em] \frac{|n|+1}{|n|} & \text{if } n \leq -1. \end{cases}$$

The shift operator $\sigma_{\mathcal{S}}$ is linearly conjugate to a skew-product of two backward weighted shifts:

$$\mathcal{F}_{\mathcal{S}} \colon \ell_p(\mathbb{R}) \times \ell_p(\mathbb{R}) \to \ell_p(\mathbb{R}) \times \ell_p(\mathbb{R})$$

$$\mathcal{F}_{\mathcal{S}}\big((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}\big) = \Big(B_\omega\left((x_n)_{n\in\mathbb{Z}}\right) + B_\omega\left((y_n)_{n\in\mathbb{Z}}\right),\, B_\omega\left((y_n)_{n\in\mathbb{Z}}\right)\Big).$$

where $\omega = \big(\omega_n((1,0))\big)_{n\in\mathbb{Z}}$ is the constant sequence equal to 1.

However, the shift operator $\sigma_{\mathcal{S}}$ is not linearly conjugate to the product of the two backward weighted shifts

$$B_{\omega\left((1,0)\right)} \times B_{\omega\left((0,1)\right)}$$

because the orbits by this product are all bounded.

This shows that our result regarding sequences $\mathcal{S} = (S_n)_{n\in\mathbb{Z}}$ with a basis $\mathcal{E}$ exhibiting $\mathcal{S}-$bounded projections is sharp.

Proof of Theorem 7

Fix $1 \leq p < +\infty$ and let $\mathcal{E} = \{b_1, \ldots, b_d\}$ be a basis of X with $\mathcal{S}-$bounded projections.

For each $n \in \mathbb{Z}$, consider the basis $\mathcal{E}_n = \{e_n(x) \colon x \in \mathcal{E}\}$ and, given $b \in \mathcal{E}$, take the linear map

$$\begin{array}{rccc} \Gamma_b \colon & \ell_p(X) & \to & \mathbb{K}^{\mathbb{Z}} \\ & (x_n)_{n \in \mathbb{Z}} & \mapsto & \left(\Pi_{e_n(b)}\,(x_n)\right)_{n \in \mathbb{Z}}. \end{array}$$

Γ_b is continuous.

Indeed, by the $\mathcal{S}-$bounded projections assumption, there exists $C > 0$ such that

$$\|\Gamma_b\big((x_n)_{n\in\mathbb{Z}}\big)\|_p^p = \sum_{n\in\mathbb{Z}} \|\Pi_{e_n(b)}(x_n)\|^p \leq C \sum_{n\in\mathbb{Z}} \|x_n\|^p = C\,\|(x_n)_{n\in\mathbb{Z}})\|_p^p.$$

This estimate also proves that

$$\Gamma_b(\ell_p(X)) \subset \ell_p(\mathbb{K}).$$

The converse inclusion, $\ell_p(\mathbb{K}) \subset \Gamma_b(\ell_p(X))$, is also true.

Indeed, given $(t_n)_{n\in\mathbb{Z}} \in \ell_p(\mathbb{K})$, the vector $\big(t_n e_n(b)\big)_{n\in\mathbb{Z}}$ belongs to $\ell_p(\mathbb{K})$ because the elements in $\mathcal{E}_n$ are normalized; and one has

$$\Gamma_b\big(\big(t_n e_n(b)\big)_{n\in\mathbb{Z}}\big) \,=\, \big(t_n\big)_{n\in\mathbb{Z}}.$$

Consider the map $\mathcal{I}\colon \ell_p(X) \to (\ell_p(\mathbb{K}))^d$ defined by

$$\begin{aligned}
\mathcal{I}\big((x_n)_{n\in\mathbb{Z}}\big) &= \big(\Gamma_{b_1}\big((x_n)_{n\in\mathbb{Z}}\big), \Gamma_{b_2}\big((x_n)_{n\in\mathbb{Z}}\big), \ldots, \Gamma_{b_d}\big((x_n)_{n\in\mathbb{Z}}\big)\big)_{n\in\mathbb{Z}} \\
&= \Big(\big(\Pi_{e_n(b_1)}(x_n)\big)_{n\in\mathbb{Z}}, \big(\Pi_{e_n(b_2)}(x_n)\big)_{n\in\mathbb{Z}}, \ldots, \big(\Pi_{e_n(b_d)}(x_n)\big)_{n\in\mathbb{Z}}\Big)
\end{aligned}$$

By construction, $\mathcal{I}$ is linear, injective and continuous.

Moreover, $\mathcal{I}$ is surjective since, given

$$\Big((\alpha_{n,1})_{n\in\mathbb{Z}}, (\alpha_{n,2})_{n\in\mathbb{Z}}, \ldots, (\alpha_{n,d})_{n\in\mathbb{Z}}\Big) \in (\ell_p(\mathbb{K}))^d$$

one has

$$\mathcal{I}\Big(\Big(\sum_{i=1}^{d} \alpha_{n,i}\, e_n(b_i)\Big)_{n\in\mathbb{Z}}\Big) = \Big((\alpha_{n,1})_{n\in\mathbb{Z}}, (\alpha_{n,2})_{n\in\mathbb{Z}}, \ldots, (\alpha_{n,d})_{n\in\mathbb{Z}}\Big)$$

and, due to the next lemma,

$$\sum_{n\in\mathbb{Z}} \Big\| \sum_{i=1}^{d} \alpha_{n,i}\, e_n(b_i)\Big\|^p \leq \sum_{n\in\mathbb{Z}} \mathcal{K}_p \sum_{i=1}^{d} |\alpha_{n,i}|^p = \mathcal{K}_p \sum_{i=1}^{d} \sum_{n\in\mathbb{Z}} |\alpha_{n,i}|^p < +\infty.$$

Lemma 13

Let X be a finite dimensional Banach space, $\mathcal{S} = (S_n)_{n \in \mathbb{Z}}$ be a sequence of linear invertible bounded operators on X and $\mathcal{E}$ be a basis of X with $\mathcal{S}-$bounded projections. Then there exists $\mathcal{K}_p > 0$ such that, for every $(\alpha_b)_{b \in \mathcal{E}} \in \mathbb{K}^{\dim X}$ and $n \in \mathbb{Z}$, one has

$$\Big\| \sum_{b \in \mathcal{E}} \alpha_b \, e_n(b) \Big\|^p \leq \mathcal{K}_p \sum_{b \in \mathcal{E}} \|\alpha_b \, e_n(b)\|^p = \mathcal{K}_p \sum_{b \in \mathcal{E}} |\alpha_b|^p.$$

Therefore, $\mathcal{I}$ is an isomorphism.

Furthermore,

$$\mathcal{I} \circ \sigma_{\mathcal{S}} = \Big(B_{\omega(b_1)} \times \cdots \times B_{\omega(b_d)} \Big) \circ \mathcal{I}$$

since, for every $b \in \mathcal{E}$,

$$\begin{aligned}
\Gamma_b\big((S_{n+1}(x_{n+1}))_{n \in Z}\big) &= \big(\Pi_{e_n(b)}\,(S_{n+1}(x_{n+1}))\big)_{n \in \mathbb{Z}} \\
&= \big(\Pi_{\frac{1}{\omega_{n+1}(b)}\,S_{n+1}(e_{n+1}(b))}\,(S_{n+1}(x_{n+1}))\big)_{n \in \mathbb{Z}} \\
&= \omega_{n+1}(b)\,\Pi_{e_{n+1}(b)}(x_{n+1}) \\
&= B_{\omega(b)}\Big(\Gamma_b\big((x_n)_{n \in \mathbb{Z}}\big)\Big)
\end{aligned}$$

and

$$
\begin{aligned}
(\mathcal{I} \circ \sigma_{\mathcal{S}})\big((x_n)_{n \in \mathbb{Z}}\big) &= \mathcal{I}\left(\big(S_{n+1}(x_{n+1})\big)_{n \in \mathbb{Z}}\right) \\
&= \Big(\Gamma_{b_1}\big((S_{n+1}(x_{n+1}))_{n \in Z}\big), \ldots, \Gamma_{b_d}\big((S_{n+1}(x_{n+1}))_{n \in Z}\big)\Big) \\
&= \Big(B_{\omega(b_1)}\big(\Gamma_{b_1}((x_n)_{n \in \mathbb{Z}})\big), \ldots, B_{\omega(b_d)}\big(\Gamma_{b_d}((x_n)_{n \in \mathbb{Z}})\big)\Big) \\
&= \big(B_{\omega(b_1)} \times \cdots \times B_{\omega(b_d)}\big)\,\big(\mathcal{I}((x_n)_{n \in Z})\big).
\end{aligned}
$$

This finishes the proof of the theorem.

The end.