Characterizing inverse sequences for which their inverse limits are homeomorphic

Matevž Črepnjak

Coauthor: Tina Sovič

University of Maribor, Slovenia

The 38th Summer Conference on Topology and its Applications
July 2024

Motivation

Mioduszewski characterized inverse sequences of polyhedra for which their inverse limits are homeomorphic (J. Mioduszewski, Mappings of inverse limits, *Colloq. Math.*, 10 (1963), 39-44.).

THEOREM 3. *If* $X = \varprojlim\{X_n, \pi_n^m\}$ *and* $Y = \varprojlim\{Y_n, \sigma_n^m\}$ *are homeomorphic, then for every sequence* $\{\varepsilon_n\}$ *such that* $\varepsilon_n > 0$ *and* $\lim \varepsilon_n = 0$, *there exists an infinite diagram*

$$(5)\qquad \begin{array}{ccccccccccc} X_{m_1} & \leftarrow & X_{m_2} & \leftarrow & \ldots & \leftarrow & X_{m_{2k-1}} & \leftarrow & X_{m_{2k}} & \leftarrow & \ldots \\ \downarrow & & \uparrow & & & & \downarrow & & \uparrow & & \\ Y_{n_1} & \leftarrow & Y_{n_2} & \leftarrow & \ldots & \leftarrow & Y_{n_{2k-1}} & \leftarrow & Y_{n_{2k}} & \leftarrow & \ldots, \end{array}$$

where $\{m_k\}$ *and* $\{n_k\}$ *are unbounded and non-decreasing sequences of positive integers, and every subdiagram of the form*

$$(5')\qquad \begin{array}{ccccc} & & X_{m_{2k-1}} & \leftarrow & X_{m_{2r}} \\ & & \downarrow & & \uparrow \\ Y_{n_i} & \leftarrow & Y_{n_{2k-1}} & \leftarrow & Y_{n_{2r}} \end{array} \qquad (5'')\qquad \begin{array}{ccccc} X_{m_i} & \leftarrow & X_{m_{2k}} & \leftarrow & X_{m_{2r-1}} \\ & & \uparrow & & \downarrow \\ & & Y_{n_{2k}} & \leftarrow & Y_{n_{2r-1}} \end{array}$$

$$(5''')\qquad \begin{array}{ccccc} X_{m_i} & \leftarrow & X_{m_{2k}} & \leftarrow & X_{m_{2r}} \\ & & \uparrow & & \uparrow \\ & & Y_{n_{2k}} & \leftarrow & Y_{n_{2r}} \end{array} \qquad (5'''')\qquad \begin{array}{ccccc} & & X_{m_{2k-1}} & \leftarrow & X_{m_{2r-1}} \\ & & \downarrow & & \downarrow \\ Y_{n_i} & \leftarrow & Y_{n_{2k-1}} & \leftarrow & Y_{n_{2r-1}} \end{array}$$

is ε_{2k}-*commutative in the cases* $(5'')$ *and* $(5''')$ *and* ε_{2k-1}-*commutative in the cases* $(5')$ *and* $(5'''')$.

Motivation

THEOREM 4. *Let* $\{\varepsilon_n\}$, $n = 1, 2, \ldots$, *be a sequence of positive numbers such that* $\lim \varepsilon_n = 0$. *The existence of an infinite diagram* (4) *having, with respect to this sequence, the properties required in Theorem* 3 *induces the existence of a homeomorphism* f *of* X *onto* Y *(the inverse of* f *is denoted by* g*) such that* $\sigma_s^{n_{2k-1}} f_k \pi_{m_{2k-1}} \underset{\varepsilon_{2k-1}}{=} \sigma_s f$ *and* $\pi_s^{m_{2k}} g_k \sigma_{n_{2k}} \underset{\varepsilon_{2k}}{=} \pi_s g$ *for every* s *and* k, $s \leqslant n_{2k-1}$ *in the first case, and* $s \leqslant m_{2k}$ *in the second one.*

THEOREM 4′. *If for every pair of positive integers* m *and* n, *for every mapping* $f_{mn}\colon X_m \to Y_n$ *belonging to* $\mathscr{F}$, *for every* $\varepsilon > 0$ *and* $m' > m$, *there exists* $n' > n$ *and a mapping* $g_{n'm'}\colon Y_{n'} \to X_{m'}$ *belonging to* $\mathscr{G}$ *such that the diagram*

$$\begin{array}{ccc} X_m & \leftarrow & X_{m'} \\ \downarrow & & \uparrow \\ Y_n & \leftarrow & Y_{n'} \end{array}$$

is ε*-commutative, and the same is true after change* X *into* Y, $\mathscr{F}$ *into* $\mathscr{G}$ *etc., then there exists a homeomorphism between* X *and* Y.

Motivation

Mioduszewski's restrictions:

- inverse limits of (not necessarily connected) polyhedra and continuous surjective bonding functions,
- functions between coordinate spaces.

Possible generalizations:

- polyhedra $\longleftrightarrow$ compact metric spaces,
- $\uparrow, \downarrow$ $\longleftrightarrow$ set-valued functions

Motivation

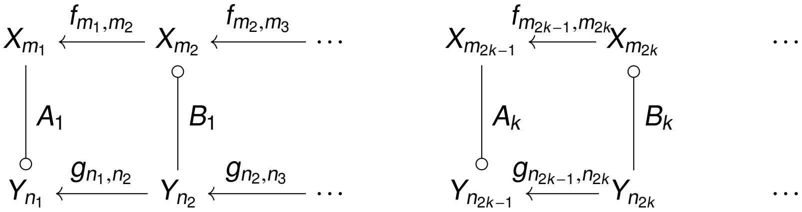
Mioduszewski's restrictions:

- inverse limits of (not necessarily connected) polyhedra and continuous surjective bonding functions,
- functions between coordinate spaces.

Possible generalizations:

- polyhedra $\longleftrightarrow$ compact metric spaces,
- $\uparrow, \downarrow \longleftrightarrow$ set-valued functions

Motivation


X_{m_1}
f_{m_1,m_2}
X_{m_2}
f_{m_2,m_3}
$\dots$
$X_{m_{2k-1}}$
$f_{m_{2k-1},m_{2k}}$
$X_{m_{2k}}$
$\dots$
A_1
B_1
A_k
B_k
Y_{n_1}
g_{n_1,n_2}
Y_{n_2}
g_{n_2,n_3}
$\dots$
$Y_{n_{2k-1}}$
$g_{n_{2k-1},n_{2k}}$
$Y_{n_{2k}}$
$\dots$

Motivation

Banič, Erceg, and Kennedy revisit Mioduszewski's results and give necessary and sufficient conditions for a compact metric space to be a continuous image of another one.

I. Banič, G. Erceg, J. Kennedy, Mappings theorem for inverse limits with setvalued bonding functions, Bull. Malays. Math. Sci. Soc., 45 (2022), 2905-2940.

Generalization

Theorem (M. Č. & T. Sovič)

Let $\{X_\ell, f_\ell\}_{\ell=1}^{\infty}$ and $\{Y_\ell, g_\ell\}_{\ell=1}^{\infty}$ be inverse sequences of compact metric spaces and surjective continuous bonding functions. Then inverse limits $\varprojlim\{X_\ell, f_\ell\}_{\ell=1}^{\infty}$ and $\varprojlim\{Y_\ell, g_\ell\}_{\ell=1}^{\infty}$ are homeomorphic if and only if there are sequences (n_k) and (m_k) of positive integers and sequences (A_k) and (B_k) of upper semicontinuous functions with surjective graphs, satisfying (1), (2), (3), (4), (5), (6), (7) or (1), (2), (3), (4), (5), (6), (8).

1. *(1)* $m_{k+1} > m_k$ *and* $n_{k+1} > n_k$ *for each positive integer* k*,*
2. *(2)* $m_{2k-1} \geq n_{2k-1}$ *and* $m_{2k} \leq n_{2k}$ *for each positive integer* k*,*
3. *(3)* $A_k : X_{m_{2k-1}} \multimap Y_{n_{2k-1}}$ *and* $B_k : Y_{n_{2k}} \multimap X_{m_{2k}}$ *for each positive integer* k*,*

Generalization

Theorem (M. Č. & T. Sovič)

Let $\{X_\ell, f_\ell\}_{\ell=1}^\infty$ and $\{Y_\ell, g_\ell\}_{\ell=1}^\infty$ be inverse sequences of compact metric spaces and surjective continuous bonding functions. Then inverse limits $\varprojlim\{X_\ell, f_\ell\}_{\ell=1}^\infty$ and $\varprojlim\{Y_\ell, g_\ell\}_{\ell=1}^\infty$ are homeomorphic if and only if there are sequences (n_k) and (m_k) of positive integers and sequences (A_k) and (B_k) of upper semicontinuous functions with surjective graphs, satisfying (1), (2), (3), (4), (5), (6), (7) or (1), (2), (3), (4), (5), (6), (8).

(1) $m_{k+1} > m_k$ *and* $n_{k+1} > n_k$ *for each positive integer* k*,*

(2) $m_{2k-1} \geq n_{2k-1}$ *and* $m_{2k} \leq n_{2k}$ *for each positive integer* k*,*

(3) $A_k : X_{m_{2k-1}} \multimap Y_{n_{2k-1}}$ *and* $B_k : Y_{n_{2k}} \multimap X_{m_{2k}}$ *for each positive integer* k*,*

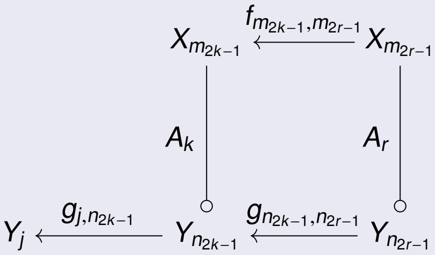
Generalization

Theorem

(4) for each positive integer k *and for each* $j \in \{1, 2, 3, \ldots, n_{2k-1}\}$,

$$g_{j,n_{2r-1}}(A_r(x)) \subseteq (g_{j,n_{2k-1}} \circ A_k \circ f_{m_{2k-1},m_{2r-1}})(x)$$

holds for each positive integer $r > k$ *and for each* $x \in X_{m_{2r-1}}$,



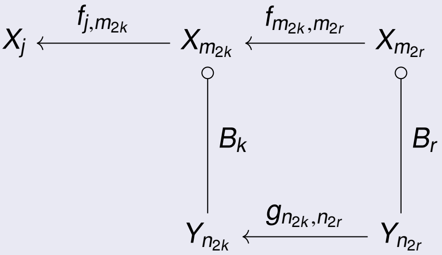
Generalization

Theorem

(5) for each positive integer k and for each $j \in \{1, 2, 3, \ldots, m_{2k}\}$,

$$f_{j,m_{2r}}(B_r(x)) \subseteq (f_{j,m_{2k}} \circ B_k \circ g_{n_{2k},n_{2r}})(x)$$

holds for each positive integer $r > k$ *and for each* $x \in Y_{n_{2r}}$,



Generalization

Theorem

(6) *for each positive integer j,*

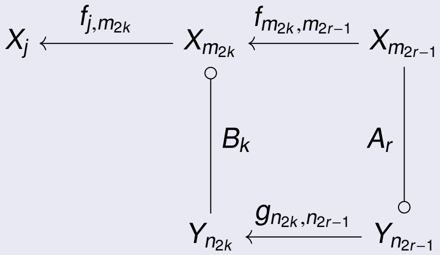
(a) $\lim_{k\to\infty} diam\Big((g_{j,n_{2k-1}} \circ A_k \circ p_{m_{2k-1}})(\boldsymbol{x})\Big) = 0$ *for each* $\boldsymbol{x} \in \varprojlim\{X_\ell, f_\ell\}_{\ell=1}^{\infty}$,

(b) $\lim_{k\to\infty} diam\Big((f_{j,m_{2k}} \circ B_k \circ q_{n_{2k}})(\boldsymbol{y})\Big) = 0$ *for each* $\boldsymbol{y} \in \varprojlim\{Y_\ell, g_\ell\}_{\ell=1}^{\infty}$,

Generalization

Theorem

(7) (a) for each positive integer k *and for each* $j \in \{1, 2, 3, \dots, m_{2k}\}$, $f_{j,m_{2r-1}}(x) \in (f_{j,m_{2k}} \circ B_k \circ g_{n_{2k},n_{2r-1}} \circ A_r)(x)$ *holds for each positive integer* $r > k$ *and for each* $x \in X_{m_{2r-1}}$,



Generalization

Theorem

(7) (b) for each positive integer j and $\varepsilon > 0$ there exist positive integers K and R, such that

$$diam\Big((f_{j,m_{2k}} \circ B_k \circ g_{n_{2k},n_{2r-1}} \circ A_r \circ p_{m_{2r-1}})(\boldsymbol{x})\Big) < \varepsilon,$$

for each $k \geq K$, $r \geq R$, $k < r$, and each $\boldsymbol{x} \in \varprojlim\{X_\ell, f_\ell\}_{\ell=1}^{\infty}$,

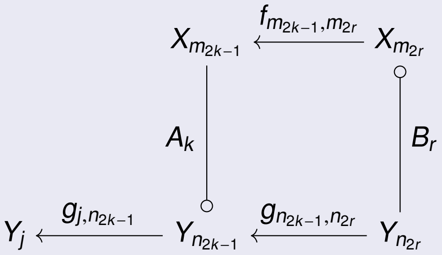
Generalization

Theorem

(8) (a) for each positive integer k and for each $j \in \{1, 2, 3, \ldots, n_{2k-1}\}$,

$$g_{j,n_{2r}}(x) \in (g_{j,n_{2k-1}} \circ A_k \circ f_{m_{2k-1},m_{2r}} \circ B_r)(x)$$

holds for each positive integer $r > k$ and for each $x \in Y_{n_{2r}}$,



Generalization

Theorem

(8) (b) for each positive integer j and $\varepsilon > 0$ there exist positive integers K and R, such that

$$diam\Big((g_{j,n_{2k-1}} \circ A_k \circ f_{m_{2k-1},m_{2r}} \circ B_r \circ q_{n_{2r}})(\mathbf{y})\Big) < \varepsilon,$$

for each $k \geq K$, $r \geq R$, $k < r$, and each $\mathbf{y} \in \varprojlim\{Y_\ell, g_\ell\}_{\ell=1}^{\infty}$.

Application

Example

Let P be the pseudo-arc and for each positive integer ℓ, let

- $X_\ell = [0,1]$, $Y_\ell = P$,
- $f_\ell : X_{\ell+1} \to X_\ell$ be any surjective function such that $\varprojlim\{X_\ell, f_\ell\}_{\ell=1}^\infty$ is a pseudo-arc, and
- $g_\ell : Y_{\ell+1} \to Y_\ell$ be the identity function on P.

Note that a continuous image of an arc is again an arc, but since the pseudo-arc contains no arcs, Banič, Erceg, and Kennedy show that for a positive $\varepsilon < \text{diam}(P)$ there are no continuous functions A_k from $X_{m_{2k-1}}$ to $Y_{n_{2k-1}}$ such that the above diagram is ε-commutative. By previous theorem there are set-valued functions A_k from $X_{m_{2k-1}}$ to $Y_{n_{2k-1}}$ with all required properties.

Application

Theorem

Let X_ℓ be a compact metric space for each positive integer ℓ and let $\sim_1$ be any equivalence relation on X_1. Further, for each positive integer ℓ, let

(1) $f_\ell : X_{\ell+1} \to X_\ell$ *be a continuous surjective function,*

(2) $\sim_{\ell+1}$ *be an equivalence relation on* $X_{\ell+1}$ *such that for each* $x, y \in X_{\ell+1}$ *it holds that* $x \sim_{\ell+1} y$ *if and only if* $f_\ell(x) = f_\ell(y)$,

(3) $g_\ell : X_{\ell+1}/_{\sim_{\ell+1}} \to X_\ell/_{\sim_\ell}$ *be defined by* $g_\ell([x]_{\ell+1}) = [f_\ell(x)]_\ell$.

(4) $\varrho_\ell : X_\ell \to X_\ell/_{\sim_\ell}$ *be the natural quotient map defined by* $\varrho_\ell(x) = [x]_\ell$ *for each* $x \in X_\ell$.

Then the inverse limits $\varprojlim\{X_\ell, f_\ell\}_{\ell=1}^\infty$ *and* $\varprojlim\{X_\ell/_{\sim_\ell}, g_\ell\}_{\ell=1}^\infty$ *are homeomorphic.*

Application

Corollary

Let (f_ℓ) be a sequence of continuous surjective functions from $[0,1]$ to $[0,1]$ such that $f_\ell(0) = f_\ell(1)$ for each positive integer ℓ. Then $\varprojlim\{[0,1], f_\ell\}_{\ell=1}^{\infty}$ is a circle-like continuum.

Example

$$f : [0,1] \to [0,1],\ f(x) = \begin{cases} 2x & ; \quad x \in [0, \frac{1}{2}] \\ 2 - 2x & ; \quad x \in [\frac{1}{2}, 1] \end{cases}$$

$\varprojlim\{[0,1], f\}_{\ell=1}^{\infty} \approx$ Brouwer–Janiszewski–Knaster continuum

Since $f(0) = f(1)$, $\varprojlim\{[0,1], f\}_{\ell=1}^{\infty}$ is circle-like.

Application

Corollary

Let (f_ℓ) be a sequence of continuous surjective functions from $[0,1]$ to $[0,1]$ such that $f_\ell(0) = f_\ell(1)$ for each positive integer ℓ. Then $\varprojlim\{[0,1], f_\ell\}_{\ell=1}^{\infty}$ is a circle-like continuum.

Example

$$f : [0,1] \to [0,1],\ f(x) = \begin{cases} 2x & ; \quad x \in [0, \frac{1}{2}] \\ 2-2x & ; \quad x \in [\frac{1}{2}, 1] \end{cases}$$

$\varprojlim\{[0,1], f\}_{\ell=1}^{\infty} \approx$ Brouwer–Janiszewski–Knaster continuum

Since $f(0) = f(1)$, $\varprojlim\{[0,1], f\}_{\ell=1}^{\infty}$ is circle-like.

Application

Corollary

Let (f_ℓ) be a sequence of continuous surjective functions from $[0,1]$ to $[0,1]$ such that $f_\ell(0) = f_\ell(1)$ for each positive integer ℓ. Then $\varprojlim\{[0,1], f_\ell\}_{\ell=1}^{\infty}$ is a circle-like continuum.

Example

$$f : [0,1] \to [0,1],\ f(x) = \begin{cases} 2x & ; \quad x \in [0, \frac{1}{2}] \\ 2-2x & ; \quad x \in [\frac{1}{2}, 1] \end{cases}$$

$\varprojlim\{[0,1], f\}_{\ell=1}^{\infty} \approx$ Brouwer–Janiszewski–Knaster continuum

Since $f(0) = f(1)$, $\varprojlim\{[0,1], f\}_{\ell=1}^{\infty}$ is circle-like.

Application

Corollary

Let (f_ℓ) be a sequence of continuous surjective functions from $[0,1]$ to $[0,1]$ such that $f_\ell(0) = f_\ell(1)$ for each positive integer ℓ. Then $\varprojlim\{[0,1], f_\ell\}_{\ell=1}^{\infty}$ is a circle-like continuum.

Example

$$f : [0,1] \to [0,1],\ f(x) = \begin{cases} 2x & ; \quad x \in [0, \frac{1}{2}] \\ 2-2x & ; \quad x \in [\frac{1}{2}, 1] \end{cases}$$

$\varprojlim\{[0,1], f\}_{\ell=1}^{\infty} \approx$ Brouwer–Janiszewski–Knaster continuum

Since $f(0) = f(1)$, $\varprojlim\{[0,1], f\}_{\ell=1}^{\infty}$ is circle-like.

Generalized inverse limits

Example

For each positive integer ℓ, let $F_\ell : [0,1] \multimap [0,1]$ be an upper semicontinuous function defined by $F_\ell(x) = [0,1]$ for each $x \in [0,1]$ and let $G_\ell : [0,1] \multimap [0,1]$ be an upper semicontinuous function defined by

$$G_\ell(x) = \begin{cases} [0,1] & ; \quad x = 0 \\ \{0\} & ; \quad x \in (0,1] \end{cases}.$$

Generalized inverse limits

Problem

Is it true that if generalized inverse sequences $\{X_\ell, F_\ell\}_{\ell=1}^\infty$ and $\{Y_\ell, G_\ell\}_{\ell=1}^\infty$ satisfy (1), (2), (3), (4), (5), (6), (7) or (1), (2), (3), (4), (5), (6), (8) in the theorem (where the bonding functions f_ℓ and g_ℓ are replaced by the set-valued functions F_ℓ and G_ℓ, respectively), then the generalized inverse limits $\varprojlim\{X_\ell, F_\ell\}_{\ell=1}^\infty$ and $\varprojlim\{Y_\ell, G_\ell\}_{\ell=1}^\infty$ are homeomorphic.

References

M. Č., T. Sovič, Characterizing inverse sequences for which their inverse limits are homeomorphic, Acta Math. Hungar., 172 (1) (2024), 42–61.

Thank you!

matevz.crepnjak@um.si