On some applications of the Steinhaus chessboard theorem

Michał Dybowski

Warsaw University of Technology
Faculty of Mathematics and Information Science

Let $n \in \mathbb{N}$. For natural $i \leq n$ we denote i*th opposite faces* of the n-dimensional unit cube I^n as

$$I^n_{i,-} = \{z \in I^n \colon z_i = 0\}, \, I^n_{i,+} = \{z \in I^n \colon z_i = 1\}.$$

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Theorem (Poincaré-Miranda)

For $n \in \mathbb{N}$ let $f \colon I^n \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ be a continuous function such that $f_i\left[I^n_{i,-}\right] \subset (-\infty, 0]$ and $f_i\left[I^n_{i,+}\right] \subset [0, \infty)$ for each natural $i \leq n$. Then, the preimage $f^{-1}\left[\{0\}\right]$ is nonempty.

We say that subset $S \subset I^n$ *connects* i*th opposite faces of* I^n if S is connected and $S \cap I^n_{i,-} \neq \emptyset \neq S \cap I^n_{i,+}$.

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Moreover we say that subset $S \subset I^n$ *connects some opposite faces of I^n* if S connects ith opposite faces of I^n for some $i \leq n$.

Theorem (W. Kulpa, L. Socha, M. Turzański)

For $n \in \mathbb{N}$ let $f\colon I^n \times I \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$ be a continuous function such that $f_i\left[I^n_{i,-} \times I\right] \subset (-\infty, 0]$ and $f_i\left[I^n_{i,+} \times I\right] \subset [0, \infty)$ for each natural $i \leq n$. Then, there exists connected subset $S \subset f^{-1}\left[\{0\}\right]$ such that $S \cap (I^n \times \{0\}) \neq \emptyset \neq S \cap (I^n \times \{1\})$.

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Reformulation

For $n \geq 2$ let $f\colon I^n \to \mathbb{R}^{n-1}$, $f = (f_1, \ldots, f_{n-1})$ be a continuous function such that $f_i\left[I^n_{i,-}\right] \subset (-\infty, p]$ and $f_i\left[I^n_{i,+}\right] \subset [p, \infty)$ for some $p \in \mathbb{R}^{n-1}$ and each natural $i \leq n-1$. Then, there exists subset $S \subset f^{-1}\left[\{p\}\right]$ which connects nth opposite faces of I^n.

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Case $n = 2$: `Let some segments of the chessboard be mined. Assume that king cannot go across the chessboard from the left edge to the right one without meeting a mined square. Then the rook can go from upper edge to the lower one moving exclusively on mined segments.`

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General case: For $n, k \in \mathbb{N}$ let $\mathcal{K}_k^n$ be the standard decomposition of I^n into k^n cubes. Let $F\colon \mathcal{K}_k^n \to \{1, \ldots, n\}$ be an arbitrary function. Then there exist $p \in \{1, \ldots, n\}$ and $\mathcal{S} \subset F^{-1}[\{p\}]$ such that $\bigcup \mathcal{S}$ connects some opposite faces of I^n.

Parametric extension of the Poincaré-Miranda Theorem

For $n \geq 2$ let $f\colon I^n \to \mathbb{R}^{n-1}$, $f = (f_1, \ldots, f_{n-1})$ be a continuous function such that $f_i\left[I^n_{i,-}\right] \subset (-\infty, p]$ and $f_i\left[I^n_{i,+}\right] \subset [p, \infty)$ for some $p \in \mathbb{R}^{n-1}$ and each natural $i \leq n-1$. Then, there exists compact subset $S \subset f^{-1}\left[\{p\}\right]$ which connects nth opposite faces of I^n.

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Theorem (D., Górka)

For $n \in \mathbb{N}$ let $f\colon I^n \to \mathbb{R}^{n-1}$ be a continuous function. Then, there exist a point $p \in \mathbb{R}^{n-1}$ and a compact subset $S \subset f^{-1}\left[\{p\}\right]$ which connects some opposite faces of the n-dimensional unit cube I^n.

Theorem

The Brouwer Fixed Point Theorem is a (simple) consequence of the result.

Proof.

Let us suppose there exists a retraction $r\colon I^n \to \partial I^n$ i.e. r is continuous and $r|_{\partial I^n}$ is the identity function.

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S1 Solving the discrete formulation of the result.

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Theorem

Let $n \in \mathbb{N}$. There exists constant $C_n > 0$ such that the following property holds:

> Let $k \in \mathbb{N}$ and $F\colon \mathcal{K}_k^n \to \mathbb{Z}^{n-1}$ be a function such that $\|F(K_1) - F(K_2)\|_\infty \leq 1$ if $K_1 \cap K_2 \neq \emptyset$. Then there exist an 1-connected subset $P \subset \mathbb{Z}^{n-1}$ with $|P| \leq C_n$ and subset $\mathcal{S} \subset F^{-1}[P]$ such that $\bigcup \mathcal{S}$ connects some opposite faces of I^n.

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The proof is based on the n-dimensional Steinhaus Chessboard Theorem and the notion of *clustered chromatic numbers* which comes from graph theory.

S2 Solving the approximate problem.

For $n \in \mathbb{N}$ let $f\colon I^n \to \mathbb{R}^{n-1}$ be a continuous function. Then, for every $\varepsilon > 0$ there exist $p_\varepsilon \in \mathbb{R}^{n-1}$ and a compact subset $S_\varepsilon \subset f^{-1}\left[\left\{B\left(p_\varepsilon, \varepsilon\right)\right\}\right]$ which connects some opposite faces of the n-dimensional unit cube I^n.

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S3 Applying the machinery of Hausdorff convergence to obtain the result from its approximate version.

Theorem

The n-dimensional Steinhaus Chessboard Theorem is a (simple) consequence of the result.

Proof.

Let $n, k \in \mathbb{N}, n \geq 2$ and $F \colon \mathcal{K}_k^n \to \{1, \ldots, n\}$ be an arbitrary function.

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We can easily see that $S \subset \bigcup F^{-1}[\{j\}]$ for some $j \leq n$, and then for $\mathcal{S}' = \{K \in F^{-1}[\{j\}] : \; S \cap K \neq \emptyset\}$, set $\bigcup \mathcal{S}'$ connects ith opposite faces of I^n since S does. $\square$

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