Construction of Hartman-Mycielski and separation axioms on semitopological groups.

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Motivation

The construction of Hartman-Mycielski associates every topological group G with a pathwise connected and locally pathwise connected topological group $G^\bullet$ and such that G can be embedded as a closed subgroup into $G^\bullet$. The existence of $G^\bullet$ brings with it various advantages, including the study of the properties of G through $G^\bullet$. Accurately, this motivates us to extend the idea of such construction in groups with different topological structures, particularly in semitopological and paratopological groups.

Definition

Let $(G, \cdot)$ be a group and τ a topology in G. Consider the following mappings:

$$\text{Given } a \in G, \quad \lambda_a : G \to G; \qquad \lambda_a(x) = ax$$

$$\rho_a : G \to G; \qquad \rho_a(x) = xa$$

$$m : G \times G \to G; \qquad m(x, y) = x \cdot y$$

$$In : G \to G; \qquad In(x) = x^{-1}$$

- *G is a semitopological group if the left and right translations, λ_a and ρ_a, are continuous.*
- *G is a paratopological group if the multiplication mapping m is continuous, when $G \times G$ is endowed with the product topology.*
- *A topological group G is a paratopological group G such that the inversion mapping In is continuous.*

Topological group	$\Rightarrow$	Paratopological group	$\Rightarrow$	Semitopological group
Semitopological group	$\not\Rightarrow$	Paratopological group	$\not\Rightarrow$	Topological group

Proposition

Every semitopological group is a homogeneous space.

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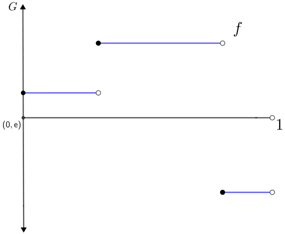
Proposition

Every semitopological group is a homogeneous space.

Let $(G,\cdot)$ be a semitopological group whose identity is denoted by e. We consider the following set:

$$G^{\bullet} = \{f\colon [0,1) \to G\colon\ f \text{ is a step function}\}.$$

It means that $G^{\bullet}$ is the set of all functions $f : J = [0,1) \to G$ such that there exists a finite sequence of real numbers $0 = a_0 < a_1 < \cdots < a_n = 1$, where the function f is constant in every $[a_k, a_{k+1})$ for all $k = 0, 1, ..., n-1$.



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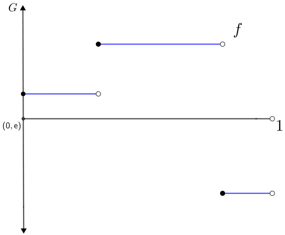
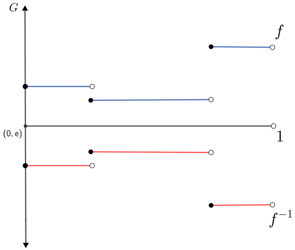


The set $G^\bullet$ is a group with the binary operation $*$ defined as: $(f * g)(r) = f(r) \cdot g(r)$ for all $r \in J$ and for any functions f and g in $G^\bullet$. The identity in $G^\bullet$ is the constant function $e^\bullet(r) = e$, for all $r \in J$, and for each function f in $G^\bullet$ its inverse is defined by $f^{-1}(r) = (f(r))^{-1}$.



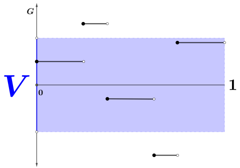
For every neighborhood V of the identity e in G and $\varepsilon > 0$, we can define the subset $O(V, \varepsilon)$ in $G^\bullet$ as follows:

$$O(V,\varepsilon) = \{f \in G^\bullet \mid \mu(\{r \in J \mid f(r) \notin V\}) < \varepsilon\},$$

where μ is the Lebesgue measure on J.

Proposition

The family of all subsets $O(V, \varepsilon)$, where V is a neighborhood of the identity e in G and ε is a positive number, forms a local base in the identity $e^\bullet$ in $G^\bullet$ making $G^\bullet$ a semitopological group.



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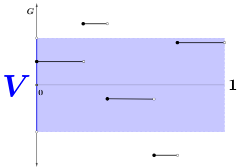

Important properties

Theorem

Every **topological group** G is topologically isomorphic to a closed subgroup of a topological group of a pathwise connected and locally pathwise connected topological group $G^\bullet$.

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The semitopological group $G^\bullet$ is pathwise connected and locally pathwise connected.

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Given a (Hausdorff) semitopological group G, then G can be embedded as a (closed) subgroup of the semitopological group $G^\bullet$.

The latter fact is false for T_1 semitopological groups.

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Example

Let $\mathbb{Z}$ be the additive group of integer numbers. We consider the topology τ on $\mathbb{Z}$ whose base consists of the following subsets:

$$\{\{k\} \cup [n, \infty) \colon k, n \in \mathbb{Z},\ k < n\}.$$

The paratopological group $(\mathbb{Z}, \tau)$ is T_1, but it is not Hausdorff. We claim that the embedding between $\mathbb{Z}$ and $\mathbb{Z}^\bullet$ is not closed. The embedding from $\mathbb{Z}$ into $\mathbb{Z}^\bullet$ is defined as follows:

$$i \colon \mathbb{Z} \to \mathbb{Z}^\bullet \quad \text{and} \quad i(x) = x^\bullet,$$

where $x^\bullet \colon [0,1) \to \mathbb{Z}$ and $x^\bullet(r) = x$ for every $r \in [0,1)$. It is shown that $i[\mathbb{Z}]$ is not a closed subgroup of $\mathbb{Z}^\bullet$. For this, it is enough to analyze the complement of $i[\mathbb{Z}]$ and verify that it is not open in $\mathbb{Z}^\bullet$.

Is the converse satisfied?.

Proposition

Let G be a semitopological group. Suppose that $i(G) \subset G^\bullet$ is closed in $G^\bullet$ then G is Hausdorff.

Theorem

Let G be a semitopological group. Then G is Hausdorff if and only if the embedding $i\colon G \to i(G) \subset G^\bullet$ is closed.

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Interaction of properties between G and $G^\bullet$

We are interested in the following properties:

- **Separation axioms**
  - T_0, T_1, Hausdorff,
  - Urysohn, semiregularity, regularity,
  - functionally Hausdorff, completely regular, Tychonoff.
- **Symmetry-like properties**
  - Almost topological groups,
  - SP-group,
  - 2-oscillating.
- **Cardinal functions**
  - Hausdorff number,
  - Symmetry number,
  - Index of regularity,
  - ω-narrowness.

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Some examples show that in paratopological groups none of the following implications hold:

$$T_0 \not\Rightarrow T_1 \not\Rightarrow T_2 \not\Rightarrow T_3.$$

In paratopological groups:

$$Regular \Rightarrow Tychonoff$$

$$semiregular \Rightarrow regular$$

In semitopological groups

It is unknown $Regular \Rightarrow Tychonoff$

$$semiregular \not\Rightarrow regular$$

Separation axioms

Let G be a semitopological group and $\mathcal{N}(e)$ a local base at the identity e in G. Then we have the following:

- G is T_0 if and only if :

$$\bigcap\{V \cap V^{-1} \colon V \in \mathcal{N}(e)\} = \{e\}.$$

- G is T_1 if and only if :

$$\bigcap_{V \in \mathcal{N}(e)} V = \{e\}.$$

- G is Hausdorff if and only if :

$$\bigcap_{V \in \mathcal{N}(e)} VV^{-1} = \{e\}.$$

Proposition

Let G be a semitopological group. Then $G^\bullet$ is Hausdorff $(T_0$ or $T_1)$ if and only if G is Hausdorff $(T_0$ or $T_1)$.

Proposition

Let G be a semitopological group. Then $G^{\bullet}$ is Urysohn (semiregular or regular) if and only if G is Urysohn (semiregular or regular).

Lemma

Let A and B be subsets of a semitopological group G. Then for any positive numbers ε and δ, it is true that:

$$O(A,\varepsilon) * O(B,\delta) \subset O(AB,\varepsilon+\delta).$$

Lemma

Let G be a semitopological group, V a subset of G and $\varepsilon > 0$. For every $0 < \delta < \varepsilon$, it follows that $\overline{O(V,\delta)} \subset O(\overline{V},\varepsilon)$.

Definition

A semitopological group G is Functionally Hausdorff if for any element $x \in G$ different of the identity e there exists a continuous function $f\colon G \to [0,1]$ such that $f(x) = 1$ and $f(e) = 0$.

For paratopological groups, the following theorem holds immediately.

Theorem

Let G be a functionally Hausdorff paratopological group. Then $G^{\bullet}$ is also a functionally Hausdorff paratopological group.

G Functionally Hausdorff	$\Longleftrightarrow$	G Hausdorff	$\Longleftrightarrow$	$G^{\bullet}$ Hausdorff	$\Longleftrightarrow$	$G^{\bullet}$ Functionally Hausdorff

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Theorem

Let d be a bounded pseudometric on a semitopological group G. For any $f, g \in G^\bullet$ consider the real-valued function $d^\bullet$ on $G^\bullet \times G^\bullet$ defined by:

$$d^\bullet(f, g) = \sum_{k=0}^{n-1} (c_{k+1} - c_k) d(x_k, y_k),$$

where the numbers $c_0, \ldots, c_n$ form a partition of both functions simultaneously and x_k, y_k are the values that f, g takes, respectively, in $[c_k, c_{k+1})$ for each $k = 0, \ldots, n-1$. Then $d^\bullet$ is a continuous bounded pseudometric on $G^\bullet$. Moreover, the number $d^\bullet(f, g)$ does not depend on the choice of the partition.

Theorem

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Paratopological case is immediate.

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Let G be a completely regular paratopological group. Then $G^\bullet$ is also a completely regular paratopological group.

For semitopological case, we use the following:

Theorem

Let G be a completely regular semitopological group. Consider an admissible uniformity $\mathcal{U}$ on G defined by a family $\mathcal{D}$ of continuous bounded pseudometrics on G. Given $f \in G^\bullet$ and a neighbourhood $O(V, \varepsilon)$ of the identity $e^\bullet$, there exist $d \in \mathcal{D}$ and a number $\delta > 0$ such that $\{g \in G^\bullet : d^\bullet(f, g) < \delta\} \subset fO(V, \varepsilon)$.

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Theorem

Let G be a completely regular semitopological group. Then $G^\bullet$ is also a completely regular semitopological group.

Corollary

Let G be a Tychonoff semitopological group. Then $G^\bullet$ is also a Tychonoff semitopological group.

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Thank you!