Remotely sequential spaces

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Definitions

A topological space X is

- compactly generated (a k-space) (Hurewicz 1948) if $A \subseteq X$ is closed if and only if $A \cap K$ is compact for every $K \subseteq X$ compact.
- sequential (Franklin 1965) if $A \subseteq X$ is not closed then there is $x \notin A$ such that $x = \lim x_n$ for some $\{x_n : n \in \omega\} \subseteq A$.
- Fréchet (Arkhangel'skii 1963) if whenever $x \in \bar{A}$ then $x = \lim x_n$ for some $\{x_n : n \in \omega\} \subseteq A$.
- remotely sequential if $A \subseteq X$ is not closed then there is some $\{x_n : n \in \omega\} \subseteq A$ $(x_n \neq x_m)$ convergent in X.

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Fréchet and sequential groups

- A large number of "pathological" consistent examples.
- (H.-Ramos García 2014) It is consistent with ZFC that every separable Fréchet group is metrizable.
- (H.-Shibakov 2022) It is consistent with ZFC that every countable sequential group is either metrizable or k_ω.

k_ω-groups

- A topological space X is k_ω if there is a countable family $\mathcal{K}$ of compact subsets of X such that a set U is open if and only if its intersection with every $K \in \mathcal{K}$ is relatively open in K.
- Countable k_ω groups are definable objects, they have $F_{\sigma\delta}$ topologies,
- Countable k_ω groups are completely classified by their compact scatteredness rank defined as the supremum of the Cantor-Bendixson index of their compact subspaces by a theorem of Zelenyuk:

Theorem (Zelenyuk 1995)

Countable k_ω groups of the same compact scatteredness rank are homeomorphic.

k_ω-groups- continued

Theorem (Zelenyuk 1995)

Countable k_ω groups of the same compact scatteredness rank are homeomorphic.

- $\alpha < \omega_1$ let $\mathcal{K}_\alpha$ be a fixed countable family of compact subsets of the rationals $\mathbb{Q}$ closed under translations, inverse and algebraic sums such that $\omega^\alpha = \sup\{\mathrm{rank}_{CB}(K) : K \in \mathcal{K}_\alpha\}$, and let

$$\tau_\alpha = \{U \subseteq \mathbb{Q} : \ \forall K \in \mathcal{K}_\alpha : U \cap K \text{ is open in } K\}.$$

- τ_0 is the discrete topology on $\mathbb{Q}$, so $\mathbb{Q}_0 = (\mathbb{Q}, \tau_0)$
- τ_α is a k_ω sequential group topology on $\mathbb{Q}$ and $\mathbb{Q}_\alpha = (\mathbb{Q}, \tau_\alpha)$
- $\mathbb{Q}$ is determined by taking into account *all* of its compact subsets, so it makes sense to denote it as $\mathbb{Q}_{\omega_1}$

Countable sequential groups - continued

Theorem (H.-Shibakov 2022)

Assuming IIA, *every countable sequential group is either metrizable or* k_ω.

Corollary

Assuming IIA, *for every infinite countable sequential group* $\mathbb{G}$ *there is exactly one* $\alpha \leqslant \omega_1$ *such that* $\mathbb{G}$ *is homeomorphic to* $\mathbb{Q}_\alpha$.

Dream conjecture

Conjecture (H.-Shibakov)

Is it consistent that every countable sequential group

1. *has a dense set without non-trivial convergent sequences,*
2. *is metrizable, or*
3. *is* k_ω.

Sequential coreflection in groups

- Given a topological space (X, τ) its sequential coreflection $(X, [\tau])$ is the strongest topology with the same convergent sequences.
- If $(\mathbb{G}, \tau)$ is a topological group and $(\mathbb{G}^2, [\tau]^2)$ is sequential then $[\tau]$ is a group topology.
- (Banakh-Zdomskyy 2004) If $(\mathbb{G}, [\tau])$ contains a copy of both the sequential fan and the convergent sequence of discrete sets then $(\mathbb{G}, \tau)$ is not remotely sequential.

Remotely sequential non-sequential spaces and groups

- There is a topology on the ordinal $\omega^\omega + 1$ which is remotely sequential and coincides with the order topology on all proper initial segments call this space X .
- The free boolean group over X is a remotely sequential non-sequential group with a k_ω-sequential coreflection.

Dream conjecture-revised

Conjecture (H.-Shibakov)

Is it consistent that every countable sequential group $(\mathbb{G}, \tau)$ *either*

1. *has a dense set without non-trivial convergent sequences,*
2. *is metrizable, or*
3. $(\mathbb{G}, [\tau])$ *is* k_ω.

We have been able to confirm the conjecture for definable spaces.

Invariant Ideal Axiom

IIA : *For every countable* *groomed* *topological group* $\mathbb{G}$ *and every* *tame, invariant, weakly closed* *ideal* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{G})$ *one of the following holds:*

1. *there is a countable* $\mathcal{S} \subseteq \mathcal{I}$ *such that for every infinite sequence* C *convergent in* $\mathbb{G}$ *there is an* $I \in \mathcal{S}$ *such that* $C \cap I$ *is infinite, (=* *countable sequence capturing)*
2. *there is a countable* $\mathcal{H} \subseteq \mathcal{I}^+$ *such that for every non-empty open* $U \subseteq \mathbb{G}$ *there is an* $H \in \mathcal{H}$ *such that* $H \setminus U \in \mathcal{I}$ *(=* *countable almost* π*-network.)*.

Open problems

1. Is it consistent (follows from IIA) that every countable group is either metrizable, has k_ω sequential coreflection or contains a dense set without a convergent subsequence?
2. Is there (in ZFC) a Fréchet group whose square is not Fréchet?
3. Is there a sequential group whose square is not sequential?

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That's all!

Thank you for your attention!