On ω-Corson and NY compact spaces

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JOINT WORK WITH A. AVILÉS

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NARODOWE CENTRUM NAUKI

Definition

A compact space K is *Eberlein compact* if K is homeomorphic to a weakly compact subset of a Banach space.

Equivalently: K is Eberlein compact iff for some Γ, K embeds into

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \forall \varepsilon > 0 \ \ \{\gamma \in \Gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\}$$

Every metrizable compactum is Eberlein compact.

Definition

A compact space K is *Corson compact* if, for some Γ, K is homeomorphic to a subset of the Σ-product of real lines

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| < \omega_1\}$$

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Here we will be interested in ω-Corson compact spaces

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The σ-product in X centered at a is

$$\sigma(X, a) = \{x \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x(\gamma) \neq a_\gamma\}| < \omega\}.$$

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Special cases:

- If $X_\gamma = [0, 1]$ and $a_\gamma = 0$ for all $\gamma \in \Gamma$, then we write $\sigma([0, 1]^\Gamma)$.
- If $X_\gamma = \mathbb{R}$ and $a_\gamma = 0$ for all $\gamma \in \Gamma$, then we write $\sigma(\mathbb{R}^\Gamma)$.
- If $X_\gamma = [0, 1]^\omega$ and $a_\gamma = (0, 0, \ldots)$ for all $\gamma \in \Gamma$, then we write $\sigma(([0, 1]^\omega)^\Gamma)$.

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- If $X_\gamma = [0,1]^\omega$ and $a_\gamma = (0,0,\ldots)$ for all $\gamma \in \Gamma$, then we write $\sigma(([0,1]^\omega)^\Gamma)$.

A compact space K is ω-Corson if K embeds into $\sigma(\mathbb{R}^\Gamma)$ for some Γ. Equivalently, K embeds into $\sigma([0,1]^\Gamma)$, for some Γ.

Remark

Every ω-Corson compact space K is *strongly countable-dimensional*, i.e. it can be written as a countable union $K = \bigcup_{n\in\mathbb{N}} K_n$ of finite-dimensional compacta K_n.

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In particular, the Hilbert cube $[0,1]^\omega$ is not ω-Corson. Because of that the class of ω-Corson compacta is rather peculiar. E.g. $\{0,1\}^\omega$ is ω-Corson and maps onto $[0,1]^\omega$ so this class is not stable under continuous images.

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A compact space K is *NY* compact if K is homeomorphic to a subset of some σ-product of metrizable compacta.

Proposition

A compact space K is *NY* compact iff K embeds into $\sigma(([0,1]^\omega)^\Gamma)$, for some Γ.

- The class $\mathcal{NY}$ of NY compacta was studied for the first time by Nakhmanson and Yakovlev in 1981.

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- Every metrizable compactum is NY compact and every NY compact space is Eberlein compact.
- The space $A(\omega_1)^\omega$, where $A(\omega_1)$ is the one-point compactification of a discrete set of size ω_1, is (uniform) Eberlein compact but not NY.

Valdivia compact spaces

Definition

A compact space K is *Valdivia* compact if there is Γ and a homeomorphic embedding $h : K \to \mathbb{R}^\Gamma$ such that $h(K) \cap \Sigma(\mathbb{R}^\Gamma)$ is dense in $h(K)$.

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Definition (Kubiś, Leiderman 2004)

A compact space K is *semi-Eberlein* if there is Γ and a homeomorphic embedding $h : K \to \mathbb{R}^\Gamma$ such that $h(K) \cap c_0(\Gamma)$ is dense in $h(K)$.

Definition

A compact space K is *NY-Valdivia* if there is Γ and an embedding $h : K \to \prod_{\gamma \in \Gamma} Q_\gamma$, where $Q_\gamma = [0,1]^\omega$ such that $h(K) \cap \sigma(([0,1]^\omega)^\Gamma)$ is dense in $h(K)$

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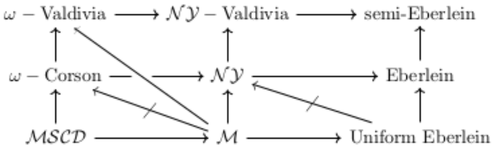
Figure: $\mathcal{MSCD}$=metrizable strongly countable-dimensional compacta; $\mathcal{M}$=metrizable compacta; $\mathcal{NY}$=NY compacta



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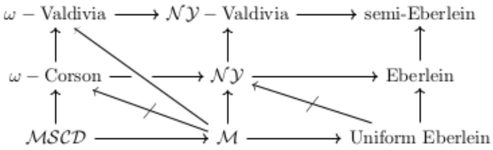

Question (Kalenda, 2022)

Does there exist an Eberlein compact space which is not NY-Valdivia?

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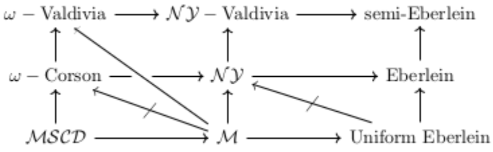

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Does there exist an Eberlein compact space which is not NY-Valdivia?

Question (Kubiś, Leiderman 2004)

Does there exist a semi-Eberlein compact space which is not ω-Valdivia?

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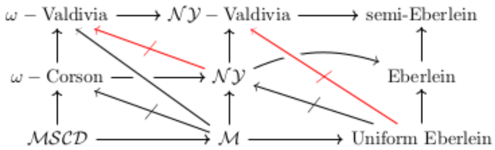

Rosenthal type characterizations

A family $\mathcal{U}$ of subsets of a topological space X is T_0-*separating* if for any $x \neq y \in X$ there is $U \in \mathcal{U}$ with $|\{x, y\} \cap U| = 1$.

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- A family $\mathcal{U}$ is *point-finite* (*point-countable*) if for any $x \in X$ the collection $\{U \in \mathcal{U} : x \in U\}$ is finite (countable).
- A family $\mathcal{U}$ is *σ-point-finite* if $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ and each $\mathcal{U}_n$ is point-finite.

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Theorem (Rosenthal)

A compact space K is Eberlein (Corson) compact iff K has a σ-point-finite (point-countable), T_0-separating family of open F_σ sets

Proposition (Marciszewski, Plebanek, Zakrzewski, 2023)

For any compact space K, TFAE:

1. K has a point-finite, T_0-separating family of open F_σ sets
2. K is scattered Eberlein compact space.

A space X is *scattered* if every nonempty subset $A \subseteq X$ has a relative isolated point.

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Theorem (Alster, 1979)

For any compact space K, TFAE:

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3. K is strong Eberlein, i.e. embeds into $\sigma(\{0,1\}^\Gamma)$

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Proposition (Marciszewski, Plebanek, Zakrzewski, 2023)

An NY compact space K is ω-Corson iff K is strongly countable-dimensional.

Theorem (Nakhmanson, Yakovlev, 1981)

For any compact space K, TFAE:

1. K is NY compact
2. K has a block-point-finite, T_0-separating family of open F_σ sets.

Let $\mathcal{M}$ be a class of separable metrizable spaces. We say that a space X is *$\mathcal{M}$-scattered* if every nonempty subset $A \subseteq X$ has a nonempty relative open subset $U \in \mathcal{M}$.

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Theorem (Marciszewski, Plebanek, Zakrzewski, 2023)

For any compact space K, TFAE:

1. K is NY compact
2. K is hereditarily metacompact and $\mathcal{M}$-scattered

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Theorem (Avilés, K., 2024)

For any compact space K, TFAE:

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2. K is Corson compact and $\mathcal{M}$-scattered
3. K is Eberlein compact $\mathcal{M}$-scattered
4. K is hereditarily metalindelöf and $\mathcal{M}$-scattered

Duplicates

Let X be a topological space. The *Alexandroff duplicate of* X is the space $AD(X)$ whose underlying set is $X \times \{0,1\}$, endowed with the following topology: Points in $X \times \{1\}$ are isolated and a basic open neighborhood of $(x,0)$ is of the form

$$(U \times \{0,1\}) \setminus \{(x,1)\},$$

where U is an open neighborhood of x in X.

It is known that K is (uniform) Eberlein compact iff $AD(K)$ is (uniform) Eberlein compact.

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2. $AD(K)$ is NY compact (ω-Corson)
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Example

$AD([0,1]^\omega)$ is NY compact but it is not ω-Valdivia

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$AD([0,1]^\omega)$ is NY compact but it is not ω-Valdivia

Question (Kubiś, Leiderman, 2004)

Does there exist a semi-Eberlein compact space which is not ω-Valdivia?

YES!

Example

Let $A(\omega_1)$ be the one-point compactification of uncountable discrete set. Consider $L = A(\omega_1)^\omega$. Then L is (uniform) Eberlein compact but not NY compact. So $AD(L)$ is (uniform) Eberlein compact but not NY-Valdivia.

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3. $AD(K)$ is semi-Eberlein

Corollary (Kubiś & Leiderman, Todorcevic)

There exists a Corson compact space which is not semi-Eberlein.

Every uniform Eberlein compact space is a continuous image of a closed subspace of the space $A(\kappa)^\omega$, where $A(\kappa)$ is the one point compactification of a discrete set of size κ.

Know:

Uniform Eberlein compacta need not be NY-Valdivia.

Question

Is every continuous image of $A(\omega_1)^\omega$ NY-Valdivia?