ON SOME IDEALS OF $\mathcal{R}L$

Mack Matlabyana

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Outline of the Talk

Preliminaries

r-ideals

z_r-ideals

Acknowledgements

A frame is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$.

A frame homomorphism is a map between frames that preserves finite meets, including the bottom element 0, and arbitrary joins, including the top element 1.

Associated with any frame homomorphism $h : L \longrightarrow M$ is its right adjoint $h_* : M \longrightarrow L$ given by

$$h_*(a) = \bigvee \{x \in L \mid h(x) \leq a\}$$

An element a is said to be rather below an element b written $a \prec b$ if there exists an element $c \in L$ such that $a \wedge c = 0$ and $b \vee c = 1$. An element a is said to be completely below an element b written $a \prec\prec b$ if there exists a sequence (c_q) indexed by the rationals $\mathbb{Q} \cap [0, 1]$ such that $c_0 = a, c_1 = b$ and $c_r \prec c_s$ whenever $r < s$.

Definition

A frame L is said to be regular if for every $a \in L$,

$$a = \bigvee\{x \in L \mid x \prec a\}.$$

Definition

A frame L is said to be completely regular if for every $a \in L$,

$$a = \bigvee\{x \in L \mid x \prec\prec a\}.$$

A set I is said to be an ideal of $\mathcal{R}L$ if

- $1 \notin I$,
- $a, b \in I$ implies that $a \vee b \in I$,
- $a \in \mathcal{R}L$ and $b \in I$ with $a \leq b$, then $a \in I$.

An ideal I of $\mathcal{R}L$ is said to be:

- *semiprime* if $\varphi \in \mathcal{R}L$ and $\varphi^2 \in I$ implies that $\varphi \in I$.
- *z-ideal* if for each $\varphi \in \mathcal{R}L$ and $\lambda \in I$, $\mathrm{coz}\varphi = \mathrm{coz}\lambda$ implies that $\varphi \in I$.
- a *radical ideal* if $I = \sqrt{I}$, where $\sqrt{I}$ is the radical of I given by $\sqrt{I} = \{\varphi \in \mathcal{R}L \mid \varphi^n \in I \text{ for some } n \in \mathbb{N}\}$.
- *d-ideal* if $Ann^2(a) \subseteq I$ for each $a \in I$.

For any frame L, we let $r(\mathcal{R}L) = \{\varphi \in \mathcal{R}L \mid \mathrm{Ann}(\varphi) = 0\}$. Then $r(\mathcal{R}L)$ is the set of all nonzero divisors of $\mathcal{R}L$, called *regular elements* of $\mathcal{R}L$. Whenever an ideal consists entirely of zero divisors, it is called a *non-regular ideal*.

Definition

An ideal I of $\mathcal{R}L$ is said to be an *r-ideal* if for each $\varphi \in \mathcal{R}L$ and $\varphi \in r(L)$, $\varphi\alpha \in I$ implies that $\varphi \in I$.

Let I be an ideal of $\mathcal{R}L$ and define

$$I_r = \{\varphi \in \mathcal{R}L \mid \varphi\alpha \in I \text{ for some } \alpha \in r(L)\}.$$

Lemma

If I is a non-regular ideal of $\mathcal{R}L$, then I_r is an r-ideal.

Lemma

Let I be a non-regular ideal of $\mathcal{R}L$. Then the following statements hold.

(i) *If I is semiprime, then I_r is also semiprime and if I is prime, then $I_r = I$.*

(ii) $Min(I_r) \subseteq Min(I)$.

(iii) *I_r is the smallest r-ideal containing I. In other words, I_r is the intersection of all r-ideals of $\mathcal{R}L$ containing I.*

(iv) *I is an r-ideal if and only if $I = I_r$.*

(v) *If I is an r-ideal, then every $P \in Min(I)$ is also an r-ideal and if I is semiprime, the converse is also true.*

(vi) *If I is a z-ideal, then I_r is also a z-ideal. The converse is not true.*

Theorem

Every d-ideal in a commutative ring R is an r-ideal.

Lemma

Let I and J be non-regular ideals of $\mathcal{R}L$. Then the following statements hold.

(i) If $I \subseteq J$, then $I_r \subseteq J_r$.

(ii) $(I \cap J)_r = I_r \cap J_r$.

(iii) $I_r + J_r \subseteq (I + J)_r$.

(iv) $I_r J_r \subseteq (IJ)_r$.

Definition

A frame L is said to be cozero complemented if for each $c \in CozL$ there is a $d \in CozL$ such that $c \wedge d = 0$ and $c \vee d$ is dense.

Proposition

A frame L is cozero complemented if and only if for each $\varphi \in \mathcal{R}L$, there is a non-zero divisor α such that $\varphi\alpha = \varphi^2$.

Proposition

The following statements are equivalent for any frame L:

(i) *L is cozero complemented.*

(ii) *Every r-ideal of $\mathcal{R}L$ is a z-ideal.*

(iii) *Every prime r-ideal of $\mathcal{R}L$ is a z-ideal.*

(iv) *For each $\varphi \in \mathcal{R}L$, $(\varphi)_r = (\varphi^2)_r$.*

Corollary

The following statements are equivalent for any frame L:

(i) *L is cozero complemented.*

(ii) *Every r-ideal of $\mathcal{R}L$ is semiprime.*

(iii) *For each ideal I of $\mathcal{R}L$, $I_z \subseteq I_r$.*

Proposition

The following are equivalent for any frame L:

(i) *L is an almost P-frame.*

(ii) *Every ideal of $\mathcal{R}L$ is an r-ideal.*

(iii) *Every ideal $I \subseteq zd(\mathcal{R}L)$ is an r-ideal.*

Proposition

The following are equivalent for any frame L:

(i) *L is an almost P-frame.*

(ii) *Every z-ideal of $\mathcal{R}L$ is an r-ideal.*

(iii) *For each ideal I of $\mathcal{R}L$, $I_r \subseteq I_z$.*

(iv) *Every prime z-ideal of $\mathcal{R}L$ is an r-ideal.*

Definition

A frame L is said to be a P-frame if for every $c \in CozL$ there is a $d \in CozL$ such that $c \wedge d = 0$ and $c \vee d = 1$.

Lemma

A frame L is a P-frame if and only if it is a cozero complemented almost P-frame.

Corollary

The following statements are equivalent:

(i) *L is a P-frame.*

(ii) *The set of all z-ideals and the set of all r-ideals of $\mathcal{R}L$ coincide.*

(iii) *For each ideal I of $\mathcal{R}L$, $I_z = I_r$*

Definition

An ideal I of $\mathcal{R}L$ is said to be a z_r-ideal if I is an r-ideal which is also a z-ideal.

Recall that for any φ in $\mathcal{R}L$, M_φ denotes the intersection of all maximal ideals of $\mathcal{R}L$ containing φ and P_φ is the intersection of all minimal prime ideals of $\mathcal{R}L$ containing φ. Now we define $(M_\varphi)_r$ by

$$\begin{aligned}(M_\varphi)_r &= \{\lambda \in \mathcal{R}L \mid \lambda\alpha \in M_\varphi \text{ for some } \alpha \in r(\mathcal{R}L)\} \\ &= \{\lambda \in \mathcal{R}L \mid \mathrm{coz}\varphi = \mathrm{coz}(\lambda\alpha) \text{ for some } \alpha \in r(\mathcal{R}L)\}.\end{aligned}$$

Lemma

An ideal I of $\mathcal{R}L$ is a z_r-ideal if and only if $(M_\varphi)_r \subseteq I$ for each $\varphi \in I$.

Corollary

An ideal I of $\mathcal{R}L$ is a z_r-ideal if and only if whenever $\varphi, \lambda \in \mathcal{R}L$ and $\alpha \in r(\mathcal{R}L)$, $coz\varphi = coz(\lambda\alpha)$ and $\varphi \in I$ implies that $\lambda \in I$.

Definition

A frame homomorphism $h : L \to M$ is said to be a C-quotient map if every $\delta : \mathcal{O}\mathbb{R} \longrightarrow L$ there is a $\gamma : \mathcal{O}\mathbb{R} \longrightarrow M$ such that $h \centerdot \delta = \gamma$. Restricting δ to bounded functions defines a C^*-quotient map.

Recall that a frame L is said to be a quasi F-frame if for every dense $a \in \mathrm{Coz}\, L$, the open quotient map $L \longrightarrow \downarrow a$ is a C^*-quotient map.

Proposition

The sum of two z_r-ideals of $\mathcal{R}L$ is a z_r-ideal or all of $\mathcal{R}L$ if and only if L is a quasi F-frame.

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