Scales and combinatorial covering properties

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Remark

We only consider subspaces of $\mathcal{P}(\omega)$.

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Definition

We say that an open cover $\mathcal{U}$ of X (such that $X \notin \mathcal{U}$) is

1. a γ-*cover*, if it is infinite and for each $x \in X$ the set $\{ U \in \mathcal{U} : x \notin U \}$ is finite;
2. an ω-*cover*, if for each finite $F \subseteq X$ there exists $U \in \mathcal{U}$, such that $F \subseteq U$.

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$\mathsf{S}_1(\mathcal{A},\mathcal{B})$: for each sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{A}$, there are sets $U_0 \in \mathcal{U}_0, U_1 \in \mathcal{U}_1, \ldots$ such that $\{\, U_n : n \in \omega \,\} \in \mathcal{B}$,

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$\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$: for each sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots \in \mathcal{A}$, there are finite sets $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots$ such that $\bigcup_{n\in\omega} \mathcal{F}_n \in \mathcal{B}$,

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Scheepers Diagram

$$
\begin{tikzcd}
 & \text{Hurewicz} & & & & \text{Menger} \\
 & \mathfrak{b}\ \mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma) \arrow[rr] & & \mathsf{S}_{\mathrm{fin}}(\Gamma,\Omega) \arrow[r] & \mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega) \arrow[r] & \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\ \mathfrak{d} \\
\mathfrak{b}\ \mathsf{S}_1(\Gamma,\Gamma) \arrow[ur] \arrow[rr] & & \mathfrak{d}\ \mathsf{S}_1(\Gamma,\Omega) \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\Gamma,\mathrm{O}) \arrow[ur] & \\
 & & & \mathsf{S}_{\mathrm{fin}}(\Omega,\Omega) \arrow[uu, crossing over] & & \\
\mathsf{S}_1(\Omega,\Gamma) \arrow[uu] \arrow[rr] & & \mathsf{S}_1(\Omega,\Omega) \arrow[uu] \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\mathrm{O},\mathrm{O})\ \mathrm{cov}(\mathcal{M}) \arrow[uu] & \\
 & & \mathrm{cov}(\mathcal{M}) & & \text{Rothberger} &
\end{tikzcd}
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Scheepers Diagram

Hurewicz property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$

Theorem

Each σ-compact space is $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

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Proof.

Since X is σ-compact, there exists a sequence of increasing compact spaces X_n, such that $X = \bigcup_{n=0}^{\infty} X_n$. Let $\mathcal{U}_0, \mathcal{U}_1, \ldots$ be a sequence of open covers of X (we assume that each cover does not contain X).

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Hurewicz's Conjecture

X is σ-compact if and only if X is $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$.

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Theorem (Just, Miller, Scheepers, Szeptycki)

Hurewicz's Conjecture is false.

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Theorem (Just, Miller, Scheepers, Szeptycki)

Hurewicz's Conjecture is false.

In the proof we consider the following two cases:

1. $\mathfrak{b} > \omega_1$;
2. $\mathfrak{b} = \omega_1$.

Definition

We say that $X = \{ x_\alpha : \alpha < \mathfrak{b} \} \subseteq [\omega]^\omega$ is a $\mathfrak{b}$-*scale*, if

1. X is unbounded with respect to $\leq^*$;
2. For all $\alpha, \beta < \mathfrak{b}$, if $\alpha < \beta$ then $x_\alpha \leq^* x_\beta$.

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Fact

There exists a $\mathfrak{b}$-scale (in ZFC).

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There exists a $\mathfrak{b}$-scale (in ZFC).

Theorem (Bartoszyński, Shelah)

Let X be a $\mathfrak{b}$-scale. Then $X \cup \mathrm{Fin}$ is Hurewicz but not σ-compact.

Rothberger property $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$

Theorem

Assume that $\mathfrak{b} \leq \mathsf{cov}(\mathcal{M})$. *Let* X *be a* $\mathfrak{b}$*-scale. Then* $X \cup \mathrm{Fin}$ *is Rothberger.*

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Theorem (Tsaban, Weiss, Bartoszyński)

Assume that $\mathfrak{b} \leq \mathrm{cov}(\mathcal{M})$. *Let* X *be a* $\mathfrak{b}$*-scale. Then* $(X \cup \mathrm{Fin})^n$ *is Rothberger for each* $n \in \omega$.

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Scheepers Diagram

Problem

Let X be a $\mathfrak{b}$-scale. Is $X \cup \mathrm{Fin}\ \mathsf{S}_1(\Gamma,\Omega)$?

Let X be a space. By $C_p(X)$ we denote the set of all continuous functions $f\colon X \to \mathbb{R}$ endowed with the topology of pointwise convergence that is the topology with the following subbase

$$S(x, U) = \{ f \in C_p(X) : f(x) \in U \}$$

where $x \in X$ and U is an open subset of $\mathbb{R}$.

Fact

Let $\{ (f_{n,m})_{m\in\omega} : n \in \omega \}$ *be a family of sequences of continuous functions such that* $(f_{n,m})_{m\in\omega}$ *converges pointwise to* 0 *for each natural* n*. If* X *is* $\mathsf{S}_1(\Gamma, \Omega)$ *then for each* n *there exists* m_n*, such that* $0 \in \mathrm{cl}\{ f_{n,m_n} : n \in \omega \}$*.*

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Theorem

Let X *be a* $\mathfrak{b}$*-scale. Then* $X \cup \mathrm{Fin}$ *is* $\mathsf{S}_1(\Gamma, \Omega)$.

Let $F \subseteq [\omega]^\omega$ be a filter. By cF we denote the filter of all cofinite subsets of ω.

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We say that $X = \{ x_\alpha : \alpha < \mathfrak{b}(F) \} \subseteq [\omega]^\omega$ is an *F-scale*, if

1. X is unbounded with respect to $\leq_F$;
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Remark

$$\mathfrak{b}\text{-}scale = \mathrm{cF}\text{-}scale$$

Theorem (Tsaban, Zdomskyy)

Let $F \subseteq [\omega]^\omega$ be a filter and X be an F-scale. Then $(X \cup \mathrm{Fin})^n$ is Menger for each $n \in \omega$.

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Theorem (Szewczak,Tsaban, Zdomskyy)

Assume that $\mathfrak{d} \leq \mathfrak{r}$ and $\mathfrak{d}$ is regular. Then there are ultrafilters $U, \tilde{U}$, U-scale X and $\tilde{U}$-scale Y such that $(X \cup \mathrm{Fin}) \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem

Let X be an F-scale and Y be $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$. Then $(X \cup \mathrm{Fin})^n \times Y$ is $\mathsf{S}_1(\Gamma, \Omega)$ for each $n \in \omega$.

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Remark

Sierpiński set is $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$.

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Corollary

Let F be a filter and X be an F-scale. Then $(X \cup \mathrm{Fin})^n$ is $\mathsf{S}_1(\Gamma, \Omega)$ for each $n \in \omega$.

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Let X be a $\mathfrak{b}$-scale. Then $(X \cup \mathrm{Fin})^n$ is $\mathsf{S}_1(\Gamma, \Omega)$ for each $n \in \omega$.

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We say that X is $\mathfrak{d}$-concentrated if $|X| \geq \mathfrak{d}$ and there exists a countable set $A \subseteq X$ such that for each open set U containing A, $|X \setminus U| < \mathfrak{d}$.

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Theorem

In the Miller model, the product of two sets that are $\mathfrak{d}$-concentrated is $\mathsf{S}_1(\Gamma, \Omega)$.

Counterexamples

Theorem

Let X be an F-scale and Y is $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$. Then $(X \cup \mathrm{Fin})^n \times Y$ is $\mathsf{S}_1(\Gamma, \Omega)$ for each $n \in \omega$.

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Let X be an F-scale and Y be $\mathsf{S}_1(\Gamma, \Gamma)$. Is $(X \cup \mathrm{Fin}) \times Y$ $\mathsf{S}_1(\Gamma, \Omega)$?

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Let X be an F-scale and Y be $\mathsf{S}_1(\Gamma, \Gamma)$. Is $(X \cup \mathrm{Fin}) \times Y$ $\mathsf{S}_1(\Gamma, \Omega)$?

We cannot replace the assumption that Y is $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$ by the assumption that Y is $\mathsf{S}_1(\Omega_{\mathrm{Bor}}, \Omega_{\mathrm{Bor}})$.

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Proposition

Assume that $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$. Then there are ultrafilters $U, \tilde{U}$, U-scale X and $\tilde{U}$-scale Y such that $X \cup \mathrm{Fin}$ and $Y \cup \mathrm{Fin}$ are $\mathsf{S}_1(\Omega_{\mathrm{Bor}}, \Omega_{\mathrm{Bor}})$ but $(X \cup \mathrm{Fin}) \times (Y \cup \mathrm{Fin})$ is not Menger.

Proposition

It is consistent with CH that there exists a set Y satisfying $\mathsf{S}_1(\Omega_{\mathrm{Bor}}, \Omega_{\mathrm{Bor}})$ and $\mathfrak{b}$-scale X such that $(X \cup \mathrm{Fin}) \times Y$ is not Menger.

κ-fin unbounded sets

Definition

We say that $X \subseteq [\omega]^\omega$ is *κ-fin-unbounded* if $|X| \geq \kappa$ and for each $d \in [\omega]^\omega$ there exits $S \subseteq X$ with $|S| < \kappa$ such that for every finite set $F \subseteq X \setminus S$ the union of F omits infinitely many intervals of d.

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Theorem

Let X be a κ-fin unbounded set where $\kappa \leq \mathfrak{d}$ and Y is $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$. Then $(X \cup \mathrm{Fin})^n \times Y$ is $\mathsf{S}_1(\Gamma, \Omega)$ for each $n \in \omega$.