Dynamics of primitive elements under group actions

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July 12, 2024

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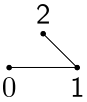
- Girth () $=\infty$,

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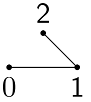
- Girth (

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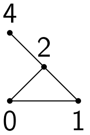
- Girth (

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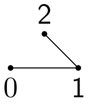
- Girth (

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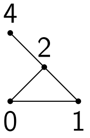
- Girth (

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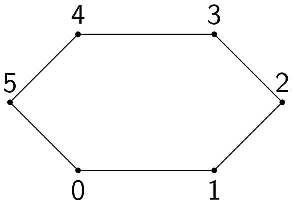
- Girth (
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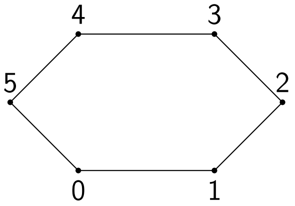
- Girth $\left(\right.$  $\left. \right)$=6,

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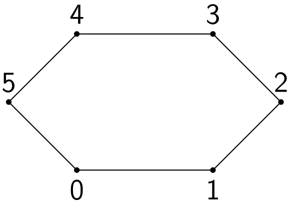
- Girth ()=6,

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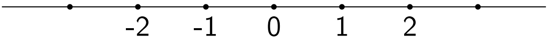
  Girth ()=∞.

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The girth of G is:

$$girth(G) = \sup_{X \in X(G)} \{girth(Cay(G, X))\}.$$

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Girth of proper HNN extensions

Theorem (Akhmedov- M., 2023)

If A, B are proper isomorphic subgroups of a finitely generated group G then girth$(\Gamma) = \infty$, where $\Gamma = (G, A, B, t)$ is the HNN extension of G with stable letter t.

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2. Use Britton's Lemma: If a word $w \in G^*_\phi$ can be expressed $w = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots t^{\epsilon_n}, n \geqslant 1$ such that w does not contain a subword of the form $t^{-1} g_i t, g_i \in A$ or $t g_j t^{-1}, g_j \in B$ then $w \neq 1$ in G^*_ϕ.

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3. Britton's Lemma establishes the existence of $\mathbb{F}_2$ for proper HNN extension.

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Key ingredients:

- Any non-elementary word hyperbolic group has infinite girth.
- Generating the ping-pong pair via the boundary action.
- For every $M > 0$, constructing a sequence of generating set S_M such that $girth(\Gamma, S_M) \geqslant M$ implying $girth(\Gamma) = \infty$.

Schreier girth

1. What is $girth(G,S) = \infty$ but $\exists\ S \sim_{\mathcal{S}} S'$ such that $girth(G,S') < \infty$?

For $S = \{s_1, \ldots, s_m\}$ an ordered generating set of a group G, one can apply the following *Schreier transformations (or Schreier moves denoted $S \sim_{\mathcal{S}} S'$)* to obtain a new generating set S' of G

- Switching s_i with s_j for some $i \neq j$,

$$S = \{s_1, s_2, \ldots, s_i, \ldots, s_j, \ldots, s_m\} \mapsto \{s_1, s_2, \ldots, s_j, \ldots, s_i, \ldots, s_m\} =$$

- Replacing s_i with s_i^{-1} for some $i \in \{1, 2, \ldots m\}$.
- Replacing s_i with $s_i s_j$ for some $i \neq j$ and $i, j \in \{1, 2, \ldots, m\}$.

Primitive elements

Definition

Fix $m \geqslant d(G)$, where $d(G)$ denotes the minimal cardinality of the generating set of G. Let $G = \langle S_0 \rangle$, be a finitely generated group, where $S_0 = \{g_1, g_2, \ldots, g_m\}$ is a fixed generating set. We define inductively the subsets $\mathcal{R}_n, n \geqslant 0$ of G as

$$\mathcal{R}_0 = \{S_0\}, \mathcal{R}_{n+1} = \{S | \langle S \rangle = G, |S| = m, \exists S' \in \mathcal{R}_n \text{ such that } S \sim_{\mathcal{S}} S'\}$$

where $S \sim_{\mathcal{S}} S'$ means S' is obtained from S by applying Schreier transformation. An element $x \in G$ is *primitive* if there exists $S \in \cup_{n \geqslant 0} \mathcal{R}_n$ such that $x \in S$.

Definition

We define the n^{th} *step primitive* elements of G as

$$\mathcal{P}_n = \{x \in G \mid x \text{ is primitive such that } \exists S \in \mathcal{R}_n \text{ with } x \in S\}$$

Examples

1. Consider the free group $\mathbb{F}_2 = \langle x, y \rangle$ with the fixed genereating set $\{x, y\}$.
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2. $\{x, y\} \sim_{\mathcal{S}} \{x, x^n y\} \sim_{\mathcal{S}} \{(x^n y)^m x, x^n y\}$.
3. The commutator $[x, y]$ is not a primitive element.

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- For all integers n, $X^n Y$ is unipotent.
- However, YXX is not unipotent.

Conjecture (Platonov and Potapchik, 2000)

For a given linear group $G = \langle S \rangle$, *with a fixed finite generating set* S, *if all the primitive elements of* G *are unipotent then is the group* G *necessarily unipotent?*

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Theorem (Platonov and Potapchik, 2000)

For any non-abelian free group $\mathbb{F}_n$, $n \geqslant 3$, let

$$\phi : Aut(\mathbb{F}_n) \to GL(m, \mathbb{C}),$$

be a finite-dimensional representation of $Aut(\mathbb{F}_n)$. Assume that all the images $\phi(x)$, where x is a primitive element of $Inn(\mathbb{F}_n)$ are unipotent and the number of Jordan blocks in $\phi(x)$ does not exceed n, then the image $\phi(Inn(\mathbb{F}_n))$ is unipotent.

Rephrasing the conjecture for group actions

For $G = \langle S \rangle$, $|S| < \infty$ and S fixed. Let G acts on some manifold X, then for any for $g \in G$, define,

$$Fix(g) = \{x \in X \mid g.x = x\}.$$

Question (1)

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Question (3)

What if $\mathcal{P}$ is replaced by $\mathcal{P}_1$ in Question (2)?

Results

(M., 2024) We study Questions (1),(2), and (3) for various group actions:

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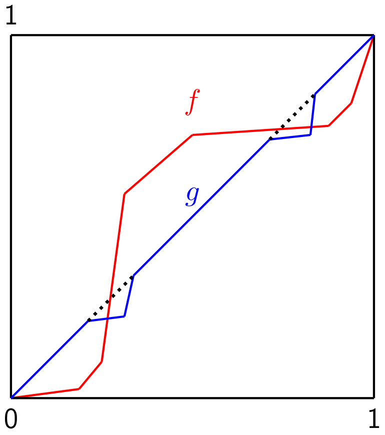
   We believe there do exist examples of non-trivial group actions answering positively Q.2. This is part of an ongoing work!

Counterexample for Q.3: $PL_+(I) \curvearrowright I$



1. All first step primitive elements fixes in $\langle f, g \rangle$ acts freely but $\langle f, g \rangle$ does not act freely.

Counterexample: $GL(2,\mathbb{R}) \curvearrowright \mathbb{R}P^1$

$$A = \begin{bmatrix} 0.13 & 0.95 \\ 0 & 1.9 \end{bmatrix}, B = \begin{bmatrix} 0.15 & 0 \\ -0.06 & 1.9 \end{bmatrix}.$$

1. Similar constructions can be arranged in higher dimensions for $GL(n+1,\mathbb{R})$ action on $\mathbb{R}P^n$!

Proposition (M.,2024)

There exists a subgroup $\Gamma \leqslant \text{Homeo}_+(I)$ such that all the primitive elements of Γ act freely but the entire group doesn't act freely.

Nielsen Girth

For a given generating set $S = \{s_1, \ldots, s_n\}$ of a group G, one can apply the following *Nielsen moves* to obtain a new generating set S' of G:

- Replacing s_k with $s_j^{\pm 1} s_k$ for some $k \neq j$ and $j, k \in \{1, 2, \ldots, n\}$ i.e.,

$$S = \{s_1, s_2, \ldots, s_k, \ldots, s_n\} \sim_N \{s_1, s_2, \ldots, s_j^{\pm} s_k, \ldots, s_n\} = S'.$$

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We write $S' \sim_N S$ to denote that S' is obtained from S by applying finitely many Nielsen move.

Definition

The k-step Nielsen girth of a group G, denoted $\mathcal{N}_k$-girth(G) is defined as,

$$\mathcal{N}_k\text{-girth}(G) = \inf_{\langle S\rangle = G, |S|=k} \left\{ \sup_{\langle S'\rangle = G, S\sim_N S'} \text{girth}\ (\text{Cay}(G, S')) \right\},$$

$k \geqslant d(G)$, where $d(G)$ is the cardinality of the minimal generating set of G.

Some immediate properties

Given $G = \langle S \rangle$, $k \geqslant d(G)$ where $d(G)$ is the cardinality of the minimal generating set of G.

1. $$\mathcal{N}_k\text{-girth}(G) = \infty \implies \mathcal{S}_k\text{-girth}(G) = \infty \implies girth(G) = \infty.$$
2. For groups satisfying a law,
$$\mathcal{N}_k\text{-girth}(G) < \infty, \mathcal{S}_k\text{-girth}(G) < \infty.$$
3. Let $N \trianglelefteq G$ such that G/N is not cyclic then
$$\mathcal{S}_k\text{-}girth(G/N) = \infty \implies girth(G) = \infty.$$

However, the converse of (1) is not true in general, as our next result gives a class of counterexamples.

Groups with $girth(G) = \infty$ but finite $\mathcal{N}_k$-girth(G)

Proposition (M.,2024)

For $n \geqslant 2$, the group Γ_n defined as

$$\Gamma_n = \langle a, b | [a, b]^n = 1 \rangle, \tag{2}$$

has finite $\mathcal{N}_2$-$girth$ but $girth(\Gamma_n) = \infty$.

Idea of proof:

1. If $S \sim_N S'$ then for any distinct $x, y \in S$ and $x', y' \in S'$ with $|S| = |S'| = 2$, we have either $[x, y] = [x', y']$ or $[x, y] = g^{-1}[x', y']g$.

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THANK YOU