Relations Between Symmetric Density and Local Symmetric Connectedness

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Definition

Let X be a set and $d : X \times X \longrightarrow [0, \infty)$ be a mapping. Then, d is a *quasi-pseudo-metric* on X if

(a) $d(x, x) = 0$ whenever $x \in X$, and

(b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We shall say that (X, d) is a T_0-*quasi-metric space* provided that d also satisfies the following condition:

For each $x, y \in X$, $d(x, y) = d(y, x) = 0$ implies that $x = y$.

Definition

Let d be a T_0-quasi-metric on a set X, then

$d^{-1} : X \times X \longrightarrow [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$

whenever $x, y \in X$, is also a T_0-quasi-metric, called the conjugate T_0-quasi-metric of d.

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If $d = d^{-1}$ then d is a metric on X.

Definition

For any T_0-quasi-metric d, also note that

$$d^s = sup\{d, d^{-1}\} = d \vee d^{-1}$$

is a metric and d^s is called the symmetrization metric of the T_0-quasi-metric d.

Incidentally, we will use the notation τ_{d^s} as the (symmetrization) topology induced by the metric d^s.

Definition

Let (X, d) be a T_0-quasi-metric space and $x \in X$.

(a)

The point $x \in X$ is called *symmetric point* if $d(x, y) = d(y, x)$ whenever $x \neq y \in X$.

(b)

The point $x \in X$ is called *antisymmetric point* if $d(x, y) \neq d(y, x)$ whenever $x \neq y \in X$.

Definition

Let (X, d) be a T_0-quasi-metric space. The pair $(x, y) \in X \times X$ is called *symmetric pair* if $d(x, y) = d(y, x)$.

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Note

For a T_0-quasi-metric space (X, d), we take

$$Z_d = \{(x, y) \in X \times X : d(x, y) = d(y, x)\}$$

as the set of symmetric pairs in (X, d). It is clear that the relation Z_d is reflexive and symmetric. Incidentally,

$$Z_d(x) = \{y \in X : (x, y) \in Z_d\}$$

is called a symmetry set of $x \in X$.

Definition

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Definition

Let (X, d) be a T_0-quasi-metric space. The pair $(x, y) \in X \times X$ is called *antisymmetric pair* if $d(x, y) \neq d(y, x)$ whenever $x \neq y$.

Definition

Given T_0-quasi-metric space (X, d), if for each $x, y \in X$, (x, y) is antisymmetric pair, then (X, d) will be called *antisymmetric space* ($d(x, y) = d(y, x) \Rightarrow x = y$ for all $x, y \in X$).

Example

On the set $\mathbb{R}$ of the reals take $u(x, y) = (x - y) \vee 0$ whenever $x, y \in \mathbb{R}$ (the standard T_0-quasi-metric on $\mathbb{R}$).
It is easy to verify that u satisfies the conditions of T_0-quasi-metric, and also $(\mathbb{R}, u)$ is an antisymmetric space.

Definition

Given (X, d) T_0-quasi-metric space, and $x, y \in X$. Let (x_i, x_{i+1}) be symmetric pairs for $i \in \{0, ..., n-1\}$. In this case, $P_{xy} = P(x = x_0, x_1, ..., x_n = y)$ is called a *symmetric path* from $x = x_0$ to $y = x_n$.

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Definition

Let (X,d) be a T_0-quasi-metric space. If for every $x,y \in X$, there exists a symmetric path P_{xy} from x to y, we say *x and y are symmetrically connected*.

Definition

The equivalence class of a point $x \in X$ with respect to the symmetric connectedness relation is called the symmetry component of x.

More clearly, if $C(x)$ denotes the symmetric connectedness relation then the symmetry component of $x \in X$ is

$$C(x) = \{y \in X : x \textit{ and } y \textit{ are symmetrically connected}\}.$$

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$$C(x) = \{y \in X : x \textit{ and } y \textit{ are symmetrically connected}\}.$$

Definition

A T_0-quasi-metric space (X, d) such that $C(x) = X$ for all $x \in X$, is called symmetrically connected.

Note.

Let (X, d) be a T_0-quasi-metric space and $C(x)$ the symmetry component of $x \in X$. If Z_d is transitive then for each $y \in C(x)$, $d(x, y) = d(y, x)$.

Note.

Let (X,d) be a T_0-quasi-metric space and $C(x)$ the symmetry component of $x \in X$. If Z_d is transitive then for each $y \in C(x)$, $d(x,y) = d(y,x)$.

Proof. Let $y \in C(x)$. Then there is a symmetric path $P_{xy}(x = x_0, x_1, ..., x_{n-1}, y = x_n)$ from x to y. Therefore $(x, x_1) \in Z_d$, $(x_1, x_2) \in Z_d$, ..., $(x_{n-1}, y) \in Z_d$. Now since Z_d is transitive then $(x, y) \in Z_d$ that is $d(x,y) = d(y,x)$.

Corollary

Each metric space is symmetrically connected.

Star Space

Let $X = [0, \infty)$ and $d : X \times X \longrightarrow [0, \infty)$

$$d(x, y) = \begin{cases} x - y & ; y \leq x \\ x + y & ; y > x \end{cases}$$

be a function on X. Thus, d is T_0-quasi-metric.

$$d^{-1} : X \times X \longrightarrow [0, \infty)$$

$$d^{-1}(x,y) = \begin{cases} y - x & ; x \leq y \\ x + y & ; x > y \end{cases}$$

is the conjugate of d T_0-quasi-metric. And

$$d^s : X \times X \longrightarrow [0, \infty)$$

$$d^s(x,y) = \begin{cases} 0 & ; \ x = y \\ x + y & ; \ x \neq y \end{cases}$$

is the symmetrization metric of d.

The balls of " 0 " on (X, d), (X, d^{-1}) and (X, d^s) are

$$B_d(0,\varepsilon) = B_{d^{-1}}(0,\varepsilon) = B_{d^s}(0,\varepsilon) = [0,\varepsilon),$$

That is, " 0 " has the same τ_{d^s}-neighborhood filter on X as the Euclidean topology.

For $0 \neq x$, we have $B_{d^s}(x,\varepsilon) = \{x\}$, where $\varepsilon > 0$. **The topology on $X \setminus \{0\}$ is discrete topology.**

In Star space " 0 " is symmetric point and for any two $x, y \in X$ there is symmetric path $P(x, 0, y)$ then Star space is symmetrically connected but it is not metric space ($d(1,3) \neq d(3,1)$).

2. Local Symmetric Connectedness in T_0-Quasi-Metric Spaces

Definition

A T_0-quasi-metric space (X, d) is called locally symmetrically connected if $C(x)$ is τ_{d^s}-open for each $x \in X$.

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Definition

A T_0-quasi-metric space (X, d) is called locally symmetrically connected if $C(x)$ is τ_{d^s}-open for each $x \in X$.

Corollary

If (X, d) is a metric space then (X, d) is locally symmetrically connected.

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A T_0-quasi-metric space (X, d) is called locally symmetrically connected if $C(x)$ is τ_{d^s}-open for each $x \in X$.

Corollary

If (X, d) is a metric space then (X, d) is locally symmetrically connected.

Lemma

Each T_0-quasi-metric space (X, d) such that τ_{d^s} is discrete is locally symmetrically connected.

Lemma

Symmetrically connected T_0-quasi-metric spaces are locally symmetrically connected.

Example

Consider the Sorgenfrey (unbounded) T_0-quasi-metric space $(\mathbb{R}, d)$ where

$$d(x,y) = \begin{cases} x - y & ; x \geq y \\ 1 & ; y > x \end{cases}$$

The conjugate of d is

$$d^{-1} : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$$

$$d^{-1}(x,y) = \begin{cases} y - x & ;\ y \geq x \\ 1 & ;\ y < x \end{cases}$$

And the symmetrization metric of d is

$$d^s : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$$

$$d^s(x,y) = \begin{cases} 0 & ;\ x = y \\ sup\{1, |x-y|\} & ;\ x \neq y \end{cases}$$

Clearly, $B_{d^s}(x, \varepsilon) = \{x\}$ for $x \in \mathbb{R}$ and $\epsilon > 0$. That is the topology τ_{d^s} is discrete.

It is easy to verify that the space $(\mathbb{R}, d)$ is not symmetrically connected, since there is no symmetric path between 1 and $3/2$.

On the other side the symmetrization topology τ_{d^s} is discrete then $(\mathbb{R}, d)$ is locally symmetrically connected since each symmetry component $C(x)$, $x \in X$ is open w.r.t the discrete topology.

Proposition

If (X, d) is a locally symmetrically connected T_0-quasi-metric space and the topological space (X, τ_{d^s}) is connected, then (X, d) is symmetrically connected.

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Proof. Note that in a locally symmetrically connected space (X, d), $C(x)$ is τ_{d^s}-clopen for each $x \in X$. Thus $C(x) = X$ by the connectedness of τ_{d^s} and so (X, d) is symmetrically connected.

3. Symmetric Density in T_0-Quasi-Metric Spaces

Definition

Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$. If for $x \in X \setminus A$, there exists $a_x \in A$ such that $d(x, a_x) = d(a_x, x)$ then A is called symmetrically-dense in (X, d).

Example

Let us define a T_0-quasi-metric p on the set $X = \{1, 2, 3\}$ via the matrix

$$P = \begin{bmatrix} 0 & 3 & 2 \\ 3 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

That is, $P = (p_{ij})$ where $p(i, j) = p_{ij}$ for $i, j \in X$. It is easy to prove that p is a T_0-quasi-metric on X.

Here note that $p(1, 2) = p(2, 1)$. Therefore, the subset $A = \{2, 3\}$ of X is symmetrically-dense in X. In addition, the subset $B = \{1\}$ is not symmetrically-dense since $p(3, 1) \neq p(1, 3)$.

Proposition

Let (X, d) be a T_0-quasi-metric space with at least two elements and $x \in X$ a symmetric point. In this case,

(a) $\{x\}$ is symmetrically-dense in X.

(b) $X \setminus \{x\}$ is symmetrically-dense in X.

Proposition

Let (X, d) be a T_0-quasi-metric space with at least two elements and $x \in X$ a symmetric point. In this case,

(a) $\{x\}$ is symmetrically-dense in X.

(b) $X \setminus \{x\}$ is symmetrically-dense in X.

Proof.

(a) By the definition of symmetric point $d(x, y) = d(y, x)$ whenever $y \in X \setminus \{x\}$ Then, $\{x\}$ is symmetrically-dense in X.

(b) Clearly, $y = x$ whenever $y \in X \setminus X \setminus \{x\}$. Thus, $d(x, a) = d(a, x)$ whenever $a \in X \setminus \{x\}$ since x is symmetric point.

Proposition

Let (X, d) be a T_0-quasi-metric space with at least two elements and $x \in X$ an antisymmetric point. Then the subsets $\{x\}$ and $X \setminus \{x\}$ cannot be symmetrically-dense in X.

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Proof. . By the definition of antisymmetric point $d(x, y) \neq d(y, x)$ whenever $y \in X \setminus \{x\}$. Then, $\{x\}$ cannot be symmetrically-dense in X.

In a similar way, clearly $y = x$ whenever $y \in X \setminus (X \setminus \{x\})$. Thus, $d(x, a) \neq d(a, x)$ whenever $a \in X \setminus \{x\}$ since x is antisymmetric point. That is, $X \setminus \{x\}$ is not symmetrically-dense. $\square$

Proposition

Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$ symmetrically-dense in X. If $A \subseteq B \subseteq X$ then B is symmetrically-dense in X.

Proof. If $x \in X \setminus B$ then $x \in X \setminus A$. Thus, by regarding the symmetric density of A, there exists $a \in A \subseteq B$ such that $d(x, a) = d(a, x)$. Then B is symmetrically-dense in X.

Example

Consider the Star space as follows: On $X = [0, \infty)$, let us take the function

$$d : X \times X \longrightarrow [0, \infty)$$

$$d(x, y) = \begin{cases} x - y & ; y \leq x \\ x + y & ; y > x \end{cases}$$

Clearly, the subset $A = \{0\} \subseteq X$ is symmetrically-dense in (X, d) since $d(x, 0) = d(0, x)$ for each $x \in X \setminus A$. Also since $A \subseteq B = \{0, 1\}$ then B is symmetrically-dense in X.

Theorem

Let (X, d) be a T_0-quasi-metric space. If $A \subseteq X$ symmetrically-dense metric subspace in X and Z_d is transitive then d is a metric.

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Let (X, d) be a T_0-quasi-metric space. If $A \subseteq X$ symmetrically-dense metric subspace in X and Z_d is transitive then d is a metric.

Proof. Take $x, y \in X$. We have four cases:

Case 1. If $x, y \in A$ then $d(x, y) = d(y, x)$.

Case 2. If $x \in A$, $y \notin A$ then there exists $a_y \in A$ such that $d(a_y, y) = d(y, a_y)$ by the symmetric density of A in X. Thus $(y, a_y) \in Z_d$. Also, we have $d(x, a_y) = d(a_y, x)$ since (A, d_A) is metric subspace. So $(x, a_y) \in Z_d$. By the fact that the relation Z_d is transitive, $(x, y) \in Z_d$. It means that $d(x, y) = d(y, x)$.

Case 3. If $x \notin A$, $y \notin A$ then there exists $a_x \in A$ and $a_y \in A$ such that $d(a_x, x) = d(x, a_x)$ and $d(a_y, y) = d(y, a_y)$ by the symmetric density of A in X. That is, $(a_x, x), (a_y, y) \in Z_d$ On the other hand, we have $(a_x, a_y) \in Z_d$ by the fact that (A, d_A) is metric subspace. Thus $d(x, y) = d(y, x)$ as the relation Z_d is transitive.

Case 4. If $y \in A$, $x \notin A$ then the proof is similar to Case 2. □

Proposition

Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$. A is symmetrically-dense in X if and only if $Z_d(x) \cap A \neq \emptyset$ for each $x \in X$.

Proposition

Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$. A is symmetrically-dense in X if and only if $Z_d(x) \cap A \neq \emptyset$ for each $x \in X$.

Proof. Take $x \in X$. There are two possibilities:

(1) If $x \in X \setminus A$, there exists $a \in A$ such that $d(x, a) = d(a, x)$ since A is symmetrically-dense in X. Thus, $a \in Z_d(x)$ and $A \cap Z_d(x) \neq \emptyset$

(2) If $x \in A$ clearly $A \cap Z_d(x) \neq \emptyset$ because of $x \in Z_d(x)$.

For the converse, take $x \in X \setminus A$. By the hypothesis, there exists $a \in X$ such that $a \in A$ and $a \in Z_d(x)$. Thus, $d(x, a) = d(a, x)$.

Proposition

Let (X, d) be a T_0-quasi-metric space and $A \subseteq X$. If $C(x) \cap A \neq \emptyset$ for each $x \in X$ and Z_d is transitive then A is symmetrically-dense in X.

Proof.

Take $x \in X \setminus A$. By the hypothesis, we have $y \in C(x) \cap A$. Thus, there is a symmetric path $P(x, x_1, ..., x_{n-1}, y)$ from x to y. Now, by the definition of symmetric path we have:

$d(x, x_1) = d(x_1, x) \to (x, x_1) \in Z_d$

$d(x_1, x_2) = d(x_2, x_1) \to (x_1, x_2) \in Z_d$

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$d(x_{n-1}, y) = d(y, x_{n-1}) \to (x_{n-1}, y) \in Z_d$.

So, by using the fact that Z_d is transitive, clearly $(x, y) \in Z_d$ and $d(x, y) = d(y, x)$.

Theorem

(X, d) be a T_0-quasi-metric space and $A \subseteq X$ symmetrically-dense in X. If the subspace (A, d_A) is symmetrically connected then (X, d) is symmetrically connected.

Proof.

Let us take $x, y \in X$ and show that there is a symmetric path from x to y.

Case 1. If $x, y \in A$ there is a symmetric path from x to y in A since (A, d_A) is symmetrically connected. It can be easily seen that there is a symmetric path from x to y in X, since $A \subseteq X$.

Case 2. If $x, y \in X \setminus A$, there are $x^{'}, y^{'} \in A$ such that $d(x, x^{'}) = d(x^{'}, x)$ and $d(y, y^{'}) = d(y^{'}, y)$ since A is symmetrically-dense in X. On the other hand, there is a symmetric path $P(x^{'}, x_1, ..., x_{n-1}, y^{'})$ in A since (A, d_A) is symmetrically connected. Now, clearly $P(x, x^{'}, x_1, ..., x_{n-1}, y^{'}, y)$ is a symmetric path from x to y in A.

Case 3. If $x \in X \setminus A$ and $y \in A$, there is $x^{'} \in A$ such that $d(x, x^{'}) = d(x^{'}, x)$ since A is symmetrically-dense in X. Also, we have a symmetric path $P(x^{'}, x_1, ..., x_{n-1}, y)$ in A from $x^{'}$ to y since (A, d_A) is symmetrically connected. Thus the path $P(x, x^{'}, x_1, ..., x_{n-1}, y)$ is also a symmetric path in X, from x to y.

Case 4. The case $y \in X \setminus A$ and $x \in A$ can be proved similar to Case 3.

Proposition

Each nonempty subset of a T_0-quasi-metric space (X, d) is symmetrically-dense in X if and only if d is metric.

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Proof. Let (X, d) be a T_0-quasi-metric space and assume that each non-empty subset of (X, d) is symmetrically-dense in X. Now take $x, y \in X$, $x \neq y$. In this case the set $A = \{x\}$ is symmetrically-dense by the hypothesis, so for $y \in X \setminus A$ there exists $a \in A$ such that $d(a, y) = d(y, a)$. Note that $a = x$, thus $d(x, y) = d(y, x)$ that is d is metric.

Conversely, suppose that d is metric and $\emptyset \neq A \subseteq X$. Then there is at least one element $a \in A$. Moreover, since (X, d) is metric space the equality $d(x, a) = d(a, x)$ will be obtained for each $x \in X \setminus A$. Finally A is symmetrically-dense in X. $\square$

There is no relation between τ_{d^s}-density and symmetric density.

Example

Let $Y = \{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N}\}$ and define $e^{'} : Y \to [0, \infty)$ as follows:

$$e^{'}(x,y) = \begin{cases} |x-y| & ; x \leq y \text{ and } (x,y) \neq (\frac{1}{2^{n+1}}, \frac{1}{2^n}) \\ 2|x-y| & ; \text{otherwise} \end{cases}$$

for $x, y \in Y$.

$e^{'}$ is a T_0-quasi-metric on Y. Also, because of the inequality $e^{'}(a,0) \neq e^{'}(0,a)$ for each $a \in Y \setminus \{0\}$ the point "0 " is antisymmetric point.

The symmetrization metric of $e^{'}$ is

$$(e^{'})^s(x,y) = \begin{cases} 0 & ; x = y \\ 2|x-y| & ; x \neq y \end{cases} \text{ for } x, y \in Y.$$

Here $(e^{'})^s(x,y) = 2|x-y| = 2m(x,y)$ for all $(x,y) \in Y \times Y$ and so $\tau_m = \tau_{2m} = \tau_{(e^{'})^s}$ on Y, where m denotes the usual metric on $\mathbb{R}$. Also, note that $m \leq e^{'} \leq (e^{'})^s$. Thus, $\tau_m = \tau_{(e^{'})}$. Since τ_m is the usual topology on $\mathbb{R}$,

$\tau_{(e^{'})^s} = \tau_{2m}$ is the discrete topology on Y except the point " 0 " .

Note that $B_{(e')^s}(0,\epsilon) = \{0, ..., \frac{1}{2^n}\}$ for $0 \in Y$ and $\epsilon > 0$. Then $Y \setminus B_{(e')^s}(0,\epsilon) = \{\frac{1}{2^{n-1}}, \frac{1}{2^{n-2}}, ..., \frac{1}{2}\}$.

The topology $\tau_{(e')^s}$ at point " 0 " is co-finite topology.

Now let us show that $V = Y \setminus \{0\}$ is $\tau_{(e')^s}$-dense. Conversely, If V is not $\tau_{(e')^s}$-dense then there is $G \in \tau_{(e')^s}$ such that $V \cap G = \emptyset$ so $G \subseteq \{0\}$ which is contradiction. Then $V = Y \setminus \{0\}$ is $\tau_{(e')^s}$-dense.

On the other side, since " 0 " is antisymmetric point, then $V = Y \setminus \{0\}$ is not symmetrically-dense.

4. Relations Between Symmetric Density and Local Symmetric Connectedness

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Proposition

Let (X,d) be a locally symmetrically connected T_0-quasi-metric space, $A \subseteq X$ and Z_d transitive. If A is τ_{d^s}-dense then A is symmetrically-dense in X.

4. Relations Between Symmetric Density and Local Symmetric Connectedness

Proposition

Let (X, d) be a locally symmetrically connected T_0-quasi-metric space, $A \subseteq X$ and Z_d transitive. If A is τ_{d^s}-dense then A is symmetrically-dense in X.

Proof. Choose $x \in X \setminus A$. Since (X, d) is locally symmetrically connected then $C(x) \in \tau_{d^s}$. Now $C(x) \cap A \neq \emptyset$ by the τ_{d^s}-density of A. Then there is $a \in X$ such that $a \in C(x) \cap A$. Therefore $a \in A$ and there is a symmetric path $P(a = x_0, x_1, ..., x_n = x)$ from a to x. By transitivity of Z_d, $d(x, a) = d(a, x)$. Thus A is symmetrically-dense. $\square$

Example

Take the Sorgenfrey T_0-quasi-metric

$$d(x,y) = \begin{cases} x - y & ; x \geq y \\ 1 & ; y > x \end{cases}$$

on $X = \{2,3\}$. As it seen before the topology τ_{d^s} is discrete then (X,d) is locally symmetrically connected. And clearly Z_d is transitive on X. Take $A = \{2\} \subseteq X$. Since $d(2,3) = d(3,2)$, then A is symmetrically-dense but since $A \cap \{3\} = \emptyset$, then A is not $\tau_{d^s|_X}$-dense.

Proposition

Let (X, d) be a T_0-quasi-metric space. If each nonempty subset of (X, d) is symmetrically-dense in X then (X, d) is locally symmetrically connected.

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Let (X, d) be a T_0-quasi-metric space. If each nonempty subset of (X, d) is symmetrically-dense in X then (X, d) is locally symmetrically connected.

Proof. Consider (X, d) is not locally symmetrically connected then it is not symmetrically connected. By the definition of symmetrically connected space there is $x \in X$ such that $C(x) \neq X$. Now let $A = X \setminus C(x)$. By the hypothesis A is symmetrically-dense. Then for $y \in X \setminus A = C(x)$ there is $a \in A$ such that $d(a, y) = d(y, a)$ then $C(y) = C(a)$. Now since $C(y) = C(x)$ (because $y \in C(x)$) then $C(a) = C(x) = C(y)$ and $a \in C(x) = X \setminus A$ which is contradiction. Then (X, d) is locally symmetrically connected. $\square$

Example

Consider Star space, that is on $X = [0, \infty)$ the function

$$d : X \times X \longrightarrow [0, \infty)$$

$$d(x, y) = \begin{cases} x - y & ; y \leq x \\ x + y & ; y > x \end{cases}$$

As previously mentioned Star space is locally symmetrically

connected but since $d(3, 5) \neq d(5, 3)$ the subset $A = \{3\}$ is not symmetrically-dense.

Theorem

Let (X, d) be a locally symmetrically connected T_0-quasi-metric space and Z_d transitive. If each nonempty subset of (X, d) is τ_{d^s} connected then d is metric.

Theorem

Let (X, d) be a locally symmetrically connected T_0-quasi-metric space and Z_d transitive. If each nonempty subset of (X, d) is τ_{d^s} connected then d is metric.

Proof. Let $x, y \in X$ and $x \neq y$. Since (X, d) is locally symmetrically connected then $C(x), C(y) \in \tau_{d^s}$. Take $A = C(x) \cup C(y) \subseteq X$. By the hypothesis A is connected then it cannot be represented as the union of two disjoint non-empty open subsets, then $C(x) \cap C(y) \neq \emptyset$. Therefore there is $z \in C(x) \cap C(y)$ and there are symmetric paths $P_{xz}(x, ..., z)$ and $P_{zy}(z, ..., y)$ then $P_{xy}(x, ..., z, ..., y)$ is a symmetric path from x to y. That is $x \in C(y)$. Now by the transitivity of Z_d, $d(x, y) = d(y, x)$ thus d is metric $\square$

Example

Let us define the metric d on the set $X = \{1, 2, 3\}$ via the matrix

$$P = \begin{bmatrix} 0 & 5 & 6 \\ 5 & 0 & 2 \\ 6 & 2 & 0 \end{bmatrix}$$

The topology $\tau_{d^s} = \tau_d$ is discrete topology on X, since the unique topology which is T_1 on a finite set is discrete topology. Then (X, d) is locally symmetrically connected and clearly Z_d is transitive. Take the subset $A = \{1, 2\}$. Now, Since $A = \{1\} \cup \{2\}$ and $\{1\} \cap \{2\} = \emptyset$ then A is not τ_{d^s}-dense.

Corollary

Let (X, d) be a locally symmetrically connected T_0-quasi-metric space and Z_d transitive. If each nonempty subset of (X, d) is τ_{d^s} connected then each nonempty subset of a T_0-quasi-metric space (X, d) is symmetrically-dense in X.

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