The topology of the polar involution of convex sets

Natalia Jonard-Pérez and Luisa F. Higueras-Montaño

Universidad Nacional Autónoma de México

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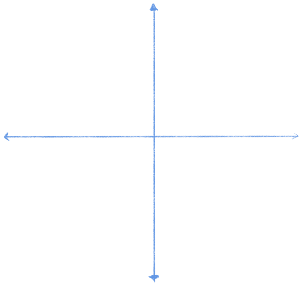
Polar Set

- Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space endowed with the standard inner product $\langle \cdot, \cdot \rangle$.
- The polar set of any nonempty subset $A \subset \mathbb{R}^n$ is defined as

$$A^\circ := \{x \in \mathbb{R}^n \mid \langle x, a \rangle \leq 1 \text{ for every } a \in A\}$$
$$= \left\{ x \in \mathbb{R}^n : \sup_{a \in A} \langle a, x \rangle \leq 1 \right\}.$$

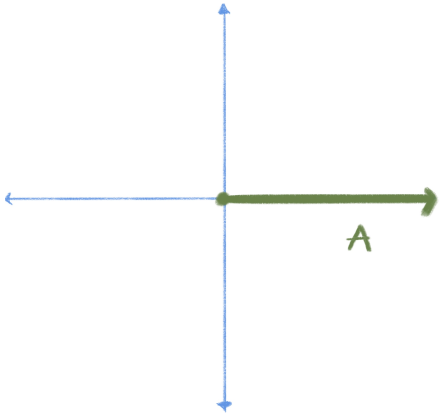
Example

- If $A = \{0\}$, then $A^{\circ} = \mathbb{R}^n$.
- If $A = \mathbb{R}^n$, then $A^{\circ} = \{0\}$.



Example

If $A = [0, \infty) \times \{0\} \subset \mathbb{R}^2$, then $A^\circ = (-\infty, 0] \times \mathbb{R}$



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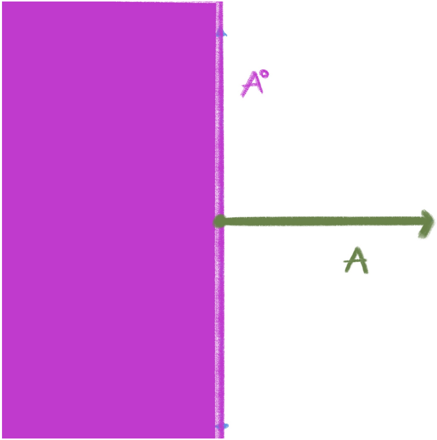
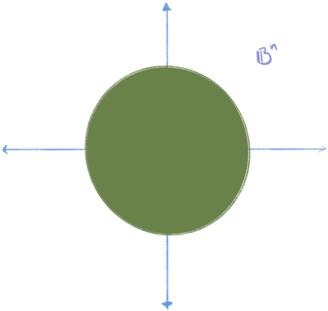


Example

If $A = \mathbb{B}^n := \{x : ||x|| \leq 1\}$, then $A^\circ = A = \mathbb{B}^n$.



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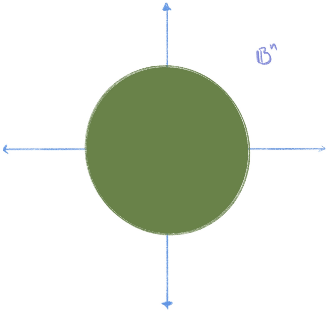

In fact,

$A^\circ = A \iff A = \mathbb{B}^n$

Basic Properties of the Polar Set

Theorem:

For every nonempty subset $A \subset \mathbb{R}^n$, the polar set A° is a closed convex subset with $0 \in A^\circ$

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Notation

- $\mathcal{K}_0^n := \{A \subset \mathbb{R}^n : A \text{ is closed, convex and } 0 \in A\}$
- $\alpha : \mathcal{K}_0^n \to \mathcal{K}_0^n$ the map given by

$$\alpha(A) = A^\circ$$

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Bipolar Theorem:

For every $A \in \mathcal{K}_0^n$, we always have that

$$(A^\circ)^\circ = A.$$

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Definition:

α is called the polar involution.

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Theorem (2008-2011):

Let $f : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a map such that for every $A, K \in \mathcal{K}_0^n$,

(D1) $f(f(A)) = A$,

(D2) $A \subseteq K$ then $f(A) \supseteq f(K)$.

Then there exists a symmetric linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

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- In 2011, B. Slomka proved the theorem as a corollary of a more general result.

- (2007-2009) S. Artstein-Avidan and V. Milman showed several examples of spaces in which all decreasing involutions are "essentially the same".

- What can we say about the polar involution from a topological point of view?

- What can we say about the polar involution from a topological point of view?
- First, we have to equip $\mathcal{K}_0^n$ with a "good topology".

The Hausdorff metric

Let (X, d) be a metric space and let $CL(X)$ be the family of all nonempty closed subsets of X.
The Hausdorff metric on $CL(X)$ is defined by

$$d_H : CL(X) \times CL(X) \to [0, \infty]$$

where

$$d_H(A, B) = \text{máx}\left\{\sup\{d(a, B) \mid a \in A\},\ \sup\{d(b, A) \mid b \in B\}\right\}.$$

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- If A or B are unbounded, then the Hausforff distance $d_H(A, B)$ can be infinite.
- $(\mathcal{K}_0^n, d_H)$ is a very ugly space with an infinite number of connected components.

The Attouch-Wets metric

The Attouch-Wets metric on $\mathcal{K}_0^n$ is defined by

$$d_{AW}(A, K) := \sup_{j \in \mathbb{N}} \left\{ \text{mín} \left\{ \frac{1}{j}, \sup_{\|x\|<j} |d(x, A) - d(x, K)| \right\} \right\}$$

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If $A, K \in K_0^n$, then

$$d_{AW}(A,K) := \sup_{j\in\mathbb{N}} \left\{ \text{mín} \left\{ \frac{1}{j}, d_H(A\cap j\mathbb{B}, K\cap j\mathbb{B}) \right\} \right\}$$

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If $A, K \in K_0^n$, then for every integer $j \geq 1$ and every $\varepsilon \in \left(\frac{1}{j+1}, \frac{1}{j}\right]$,

$$d_{AW}(A,K) < \varepsilon \iff d_H\big(A\cap j\mathbb{B}, K\cap j\mathbb{B}\big) < \varepsilon.$$

Remark:

The topology generated by d_{AW} on $\mathcal{K}_0^n$ coincides with the Fell topology and the Wijsman topology.

Fell topology

The Fell topology on $\mathcal{K}_0^n$ is the topology generated by the sets

$$(\mathbb{R}^n \setminus K)^+ := \{A : A \cap K = \emptyset\}$$

$$U^- := \{A : A \cap U \neq \emptyset\}$$

where $K \subset \mathbb{R}^n$ is compact and $U \subset \mathbb{R}^n$ is open.

The Wijsman topology

The Wijsman topology on $\mathcal{K}_0^n$ is the topology generated by the sets

$$U(x,\varepsilon)^+ := \{A : d(x,A) < \varepsilon\},$$

$$U(x,\varepsilon)^- := \{A : d(x,A) > \varepsilon\}.$$

Who is $(\mathcal{K}_0^n, d_{AW})$?

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Definition:

The Hilbert cube is the topological product

$$Q := \prod_{n \in \mathbb{N}} [-1, 1].$$

Topological classification of some families of convex sets

- $(\mathcal{K}_b^n, d_H) \cong Q \times [0,1)$, where $\mathcal{K}_b^n$ denotes the family of all compact convex subsets of $\mathbb{R}^n$ (S. Nadler, J. Quinn y N. Stavrakas, 1979).

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- $(\mathcal{K}_{(b)}^n, d_H) \cong Q \times \mathbb{R}^{\frac{n(n+3)}{2}}$, where $\mathcal{K}_{(b)}^n$ denotes the family of all compact convex subsets of $\mathbb{R}^n$ with non empty interior (S. Antonyan y N. J-P, 2013).

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The Hausdorff metric and the Attouch-Wets metric are equivalent in $\mathcal{K}_b^n$.

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Theorem (L. F. Higueras-Montaño and N. J.-P.)

$(\mathcal{K}_0^n, d_{AW})$ is homeomorphic with the Hilbert cube.

Theorem (R. A. Wijsman, 1963 + some recent remarks)

$\alpha : (\mathcal{K}_0^n, d_{AW}) \to (\mathcal{K}_0^n, d_{AW})$ is continuous.

Let us recall that...

For every $A, B \in \mathcal{K}_0^n$ the following hold:

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Remark:

The polar map is a continuous involution on $\mathcal{K}_0^n$ (which is homeomorphic to the Hilbert cube) with a unique fixed point.

Standard Involution on Q

Let Q be the Hilbert cube and $\sigma : Q \to Q$ be defined by

$$\sigma(x) = -x.$$

- The map σ is a continuous involution with a unique fixed point.
- The involution σ is called the standard involution on Q.

Anderson's Problem (1960's)

Open Problem

If $\beta : Q \to Q$ is a continuous involution with a unique fixed point, does there exist a homeomorphism $\Psi : Q \to Q$ such that $\beta = \Psi^{-1}\sigma\Psi$?

$$\begin{array}{ccc} Q & \xrightarrow{\beta} & Q \\ {\scriptstyle\Psi}\downarrow & & \uparrow{\scriptstyle\Psi^{-1}} \\ Q & \xrightarrow{\sigma} & Q \end{array}$$

In other words: is σ topologically conjugate to β?

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Theorem (J. van Mill and J. West, 2020)

Let $\sigma_{\mathbb{R}^\infty} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ and $\sigma_{\ell_2} : \ell_2 \to \ell_2$ be defined as

$$\sigma_{\mathbb{R}^\infty}(x) = -x \qquad \sigma_{\ell_2}(x) = -x.$$

Even if $\mathbb{R}^\infty$ and ℓ_2 are homeomorphic, the involutions $\sigma_{\mathbb{R}^\infty}$ and σ_{ℓ_2} are not topologically conjugate to each other.

Theorem 1 (L. F. Higueras-Montaño y N. J-P., 2022):

The polar involution $\alpha : \mathcal{K}_0^n \to \mathcal{K}_0^n$ is topologically conjugate to the standard involution $\sigma : Q \to Q$.

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Theorem (2008-2011):

Let $f : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be a map such that for every $A, K \in \mathcal{K}_0^n$,

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- From an algebraic point of view all decreasing involutions on $\mathcal{K}_0^n$ are equivalent.
- From a dynamical point of view this is not true!

Theorem 2 (L. F. Higueras-Montaño and N. J-P., 2022):

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a symmetric linear isomorphism and let $f : \mathcal{K}_0^n \to \mathcal{K}_0^n$ be defined as $f(A) = T(A^\circ)$. Then, the following statements hold.

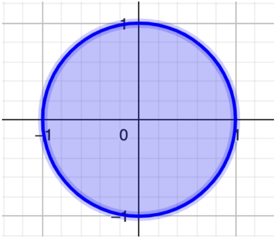
1. If T is positive-definite, then f is conjugate with the polar mapping. In particular, f has a unique fixed point.
2. If T is not positive-definite, then f has infinitely many fixed points.

Example

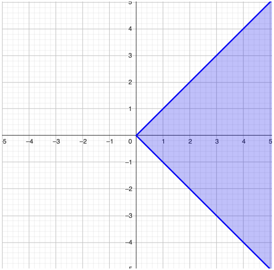
Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear isomorphism given by $T(x, y) = (-x, y)$ and $f : \mathcal{K}_0^2 \to \mathcal{K}_0^2$ be defined by

$$f(A) = T(A^\circ)$$

Then the following elements of $\mathcal{K}_0^2$ are fixed points of f:



$\{(x, y) : x^2 + y^2 \le 1\}$



$\{(x, y) : x \ge |y|\}$

Corollary

Every decreasing involution $f : \mathcal{K}_0^n \to \mathcal{K}_0^n$ with a unique fixed point is conjugate with the standard involution on Q. Moreover, f is of the form $f(A) = T(A^\circ)$ for some positive-definite linear isomorphism $T : \mathbb{R}^n \to \mathbb{R}^n$.

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Namely:

Every decreasing involution $f : \mathcal{K}_0^n \to \mathcal{K}_0^n$ with a unique fixed point is the polar map with respect to some inner product on $\mathbb{R}^n$.

Again, these properties!!!

For every $A, B \in \mathcal{K}_0^n$ the following hold:

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These three properties characterize the polar involution(s) on $\mathcal{K}_0^n$

Final remarks and questions

Remark:

- The order given by the inclusion $\subset$ on $\mathcal{K}_0^n$ defines a lattice structure, where the operations $\wedge$ and $\vee$ are given by

$$K \wedge L := K \cap L, \qquad K \vee L := \overline{\text{conv}(K \cup L)}.$$

Final remarks and questions

- The Hilbert cube $Q = \prod_{n\in\mathbb{N}}[-1,1]$ has a lattice structure where the order $\preceq$ is defined by

$$x \preceq y \iff x_n \leq y_n \text{ for every } n \in \mathbb{N},$$

and the operations $\vee$ and $\wedge$ are defined by

$$x \vee y := (\text{máx}\{x_i, y_i\})_i \qquad x \wedge y := (\text{mín}\{x_i, y_i\})_i.$$

Final remarks and questions

- The Hilbert cube $Q = \prod_{n\in\mathbb{N}}[-1,1]$ has a lattice structure where the order $\preceq$ is defined by

$$x \preceq y \iff x_n \leq y_n \text{ for every } n \in \mathbb{N},$$

and the operations $\vee$ and $\wedge$ are defined by

$$x \vee y := (\text{máx}\{x_i, y_i\})_i \qquad x \wedge y := (\text{mín}\{x_i, y_i\})_i.$$

- The standard involution is decreasing with respect to $\preceq$.

Final remarks and questions

After Slomka's result (c.f. S. Arstein-Avidan and V. Milman), it is natural to ask the following weak version of Anderson's problem:

Question (Anderson's problem):

Let $\beta : Q \to Q$ be a continuous involution with a unique fixed point, Is β conjugate with the standard involution?

Final remarks and questions

After Slomka's result (c.f. S. Arstein-Avidan and V. Milman), it is natural to ask the following weak version of Anderson's problem:

Question (Weak version of Anderson's problem):

Let $\beta : Q \to Q$ be a continuous involution with a unique fixed point. Assume that there exists a lattice structure $(\preceq, \wedge, \vee)$ such that β is decreasing with respect to $\preceq$.
Is β conjugate with the standard involution?

Final remarks and questions

- (A. López Poo) If the lattice is modular (namely, if $a \preceq b$ implies that $a \wedge (x \vee b) = (a \wedge x) \vee b$) and the operations $\vee$ and $\wedge$ are continuous (such as it happens with the natural lattice structure on Q), then the answer is yes.

Final remarks and questions

- (A. López Poo) If the lattice is modular (namely, if $a \preceq b$ implies that $a \wedge (x \vee b) = (a \wedge x) \vee b$) and the operations $\vee$ and $\wedge$ are continuous (such as it happens with the natural lattice structure on Q), then the answer is yes.
- Notice that the operations $\vee$ and $\wedge$ are not continuous on $\mathcal{K}_0^n$ and the lattice structure is not modular.

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Thank you!