An Algorithm to Detect Non-Order Preserving Braids

Joint with J. Johnson and H. Turner

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SUMTOPO 2024

What is a *biorder* on a group?

When is a group *biorderable*?

Definition:

A ***biorder*** on a group G is a strict total ordering that is invariant under **both** left and right "multiplication".

A group is ***biorderable*** if there exists a biordering of its elements.

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Example: The integers are biorderable.

Addition is a translation

…-3 -2 -1 0 1 2 3…

non-Example: Torsion is not biorderable.

$\mathbb{Z}_4$

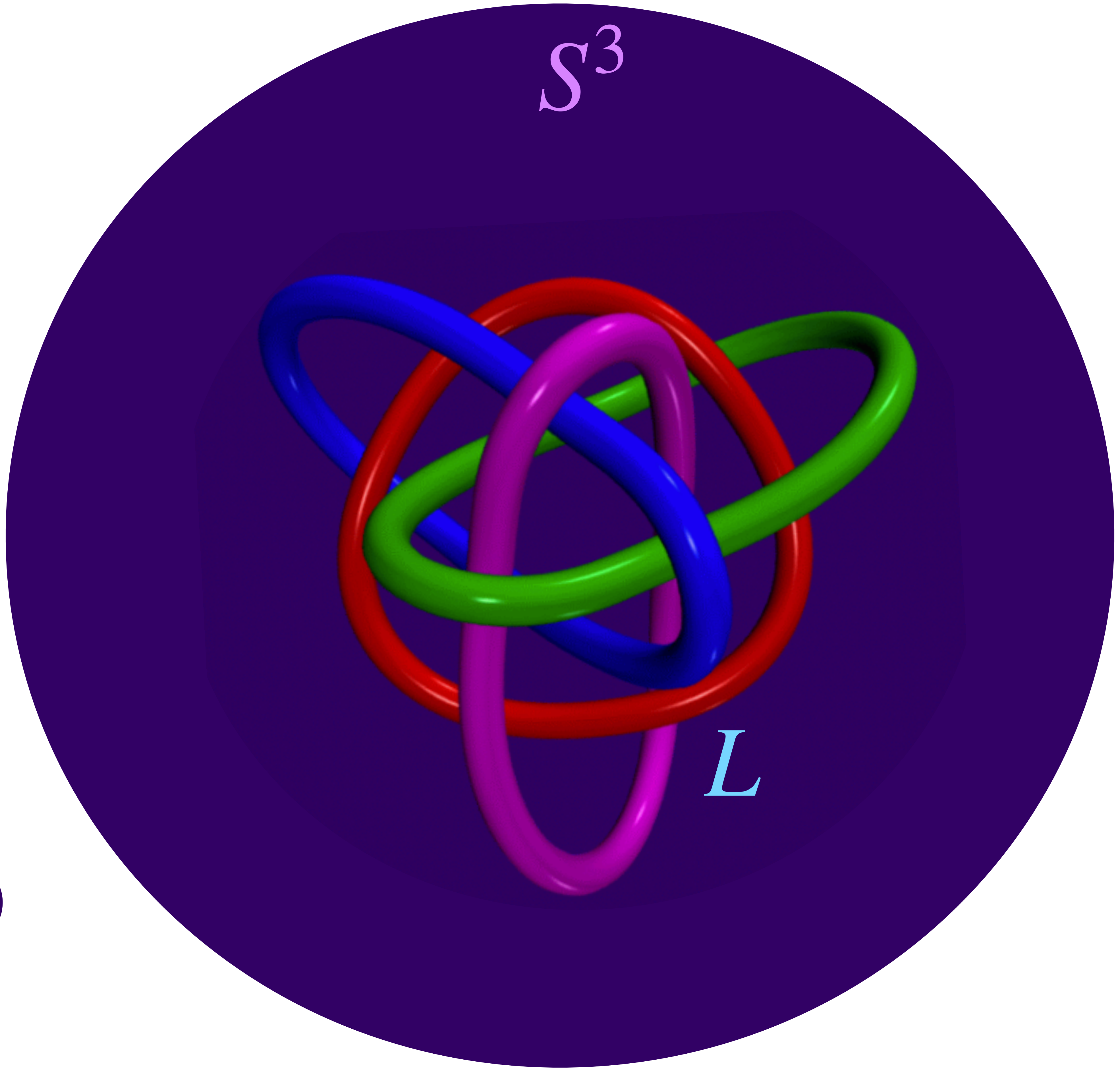
0 < 1 < 2 < 3

2+0 < 2+1 < 2+2 < 2+3

2 < 3 < 0 < 1

Why bi-order a group?

Motivation from 3-mfd topology



$$\pi_1(S^3 - L) = \pi_1(L)$$

Boyer-Rolfsen-Wiest (2005):

$\pi_1(L)$ is left orderable.

Say "L is left-orderable"

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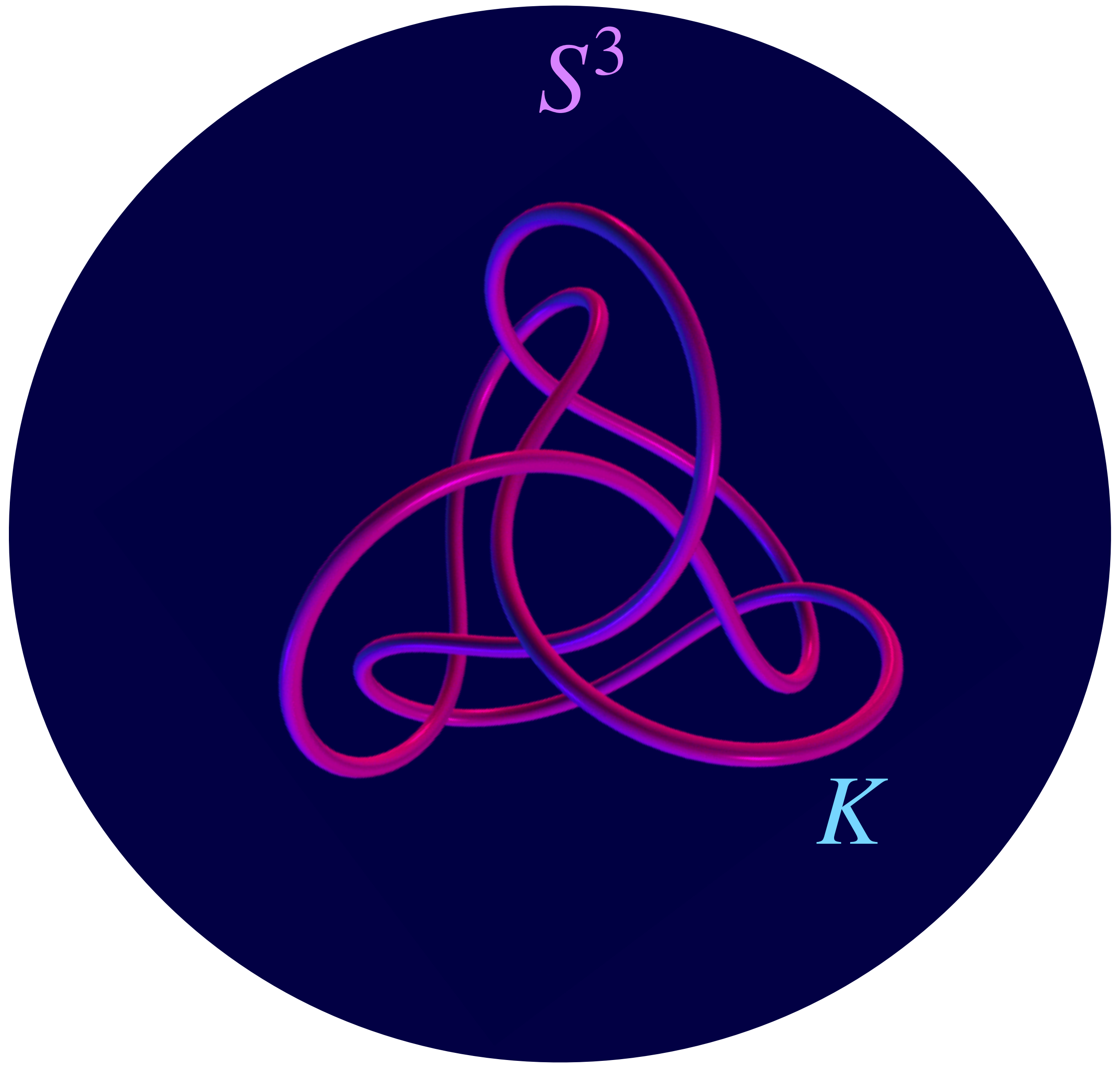
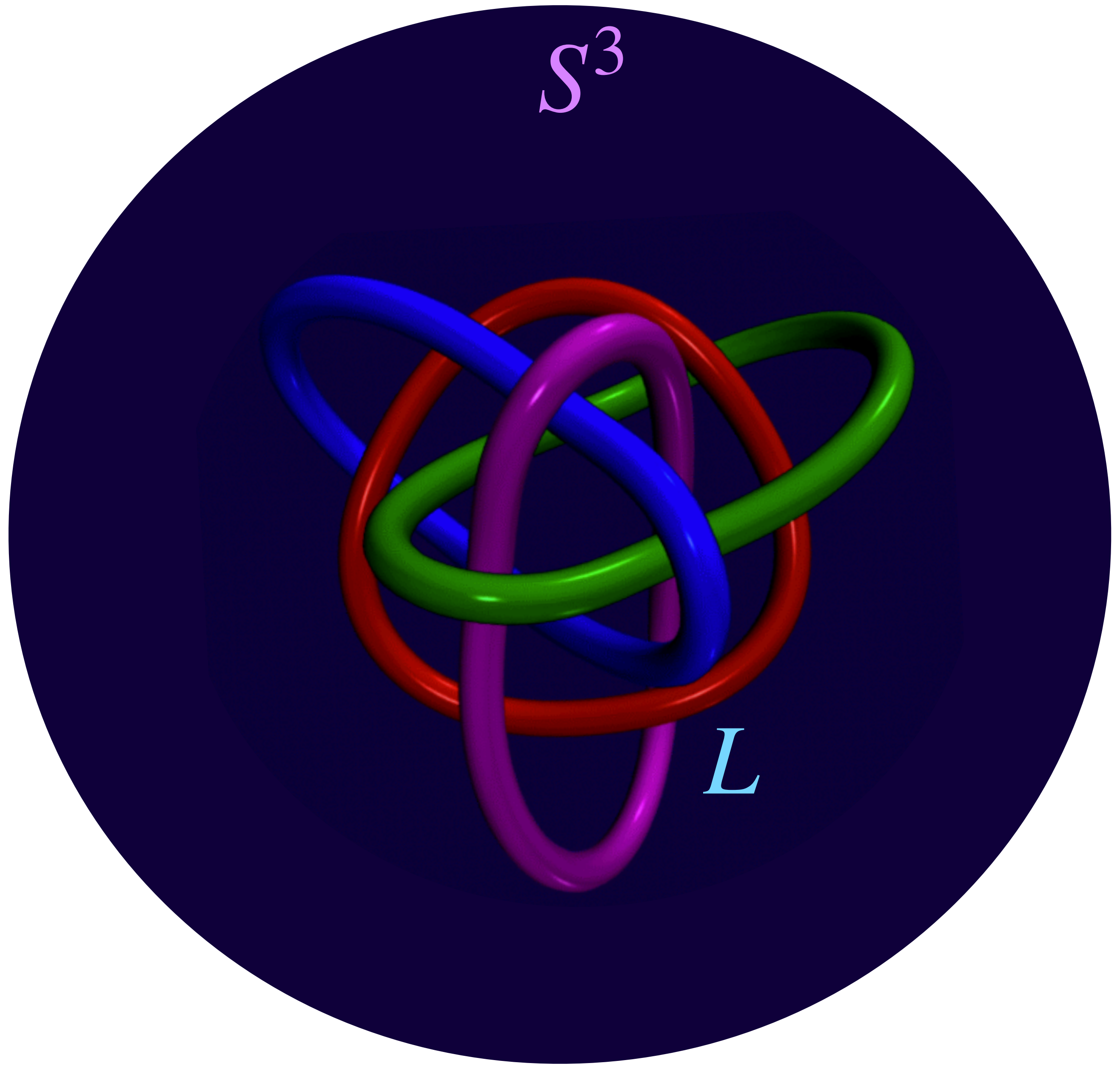


Knots vs Links



Clay-Rolfsen (2012):

$\pi_1(K)$ is biorderable

$\Rightarrow$ K is NOT an L-space knot

(No nontrivial surgery is an L-space)

There exists links that are biorderable AND L-space links.

(Complement BO but you get a new space that is NOT left orderable by L-Conj)

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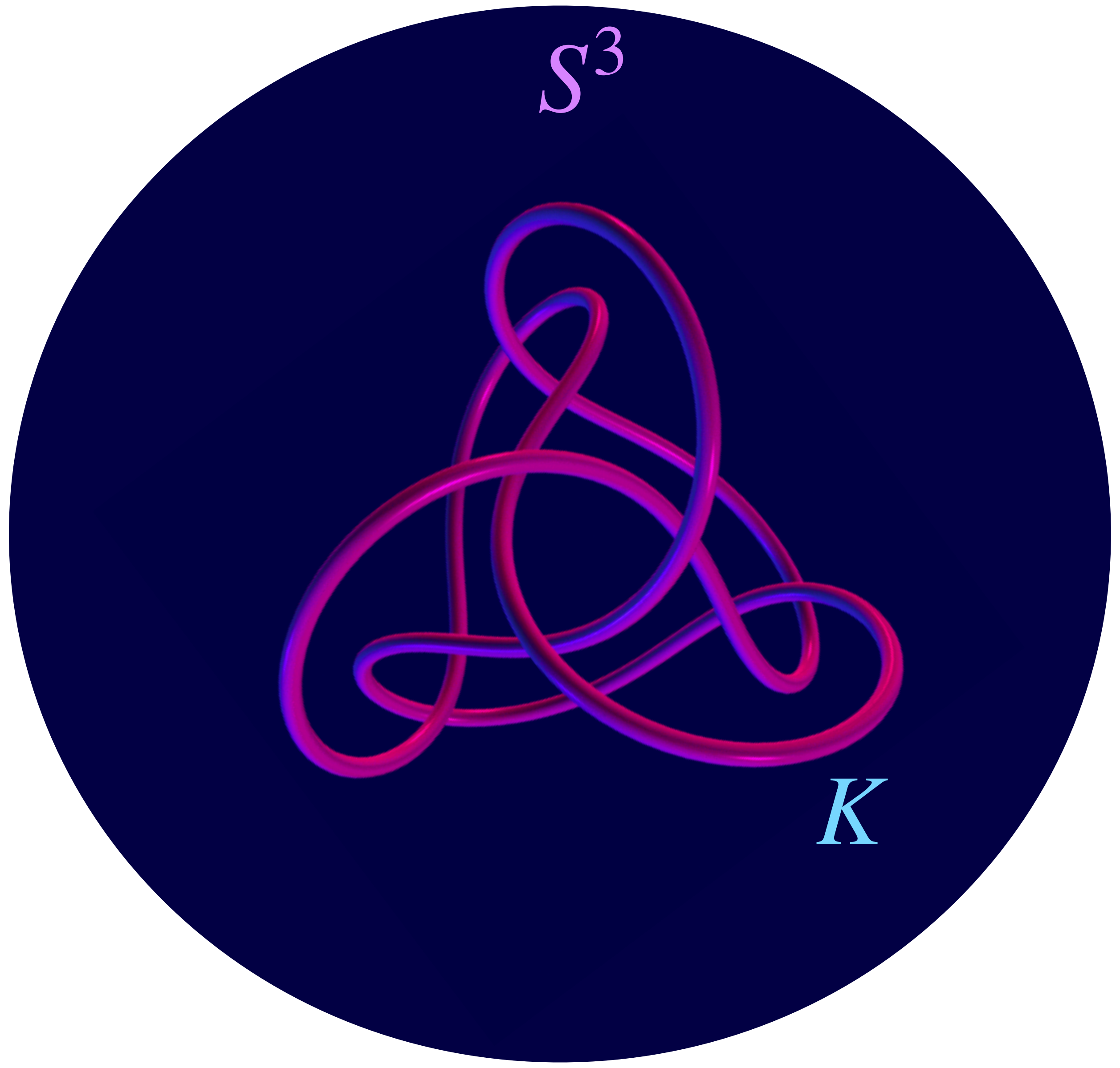

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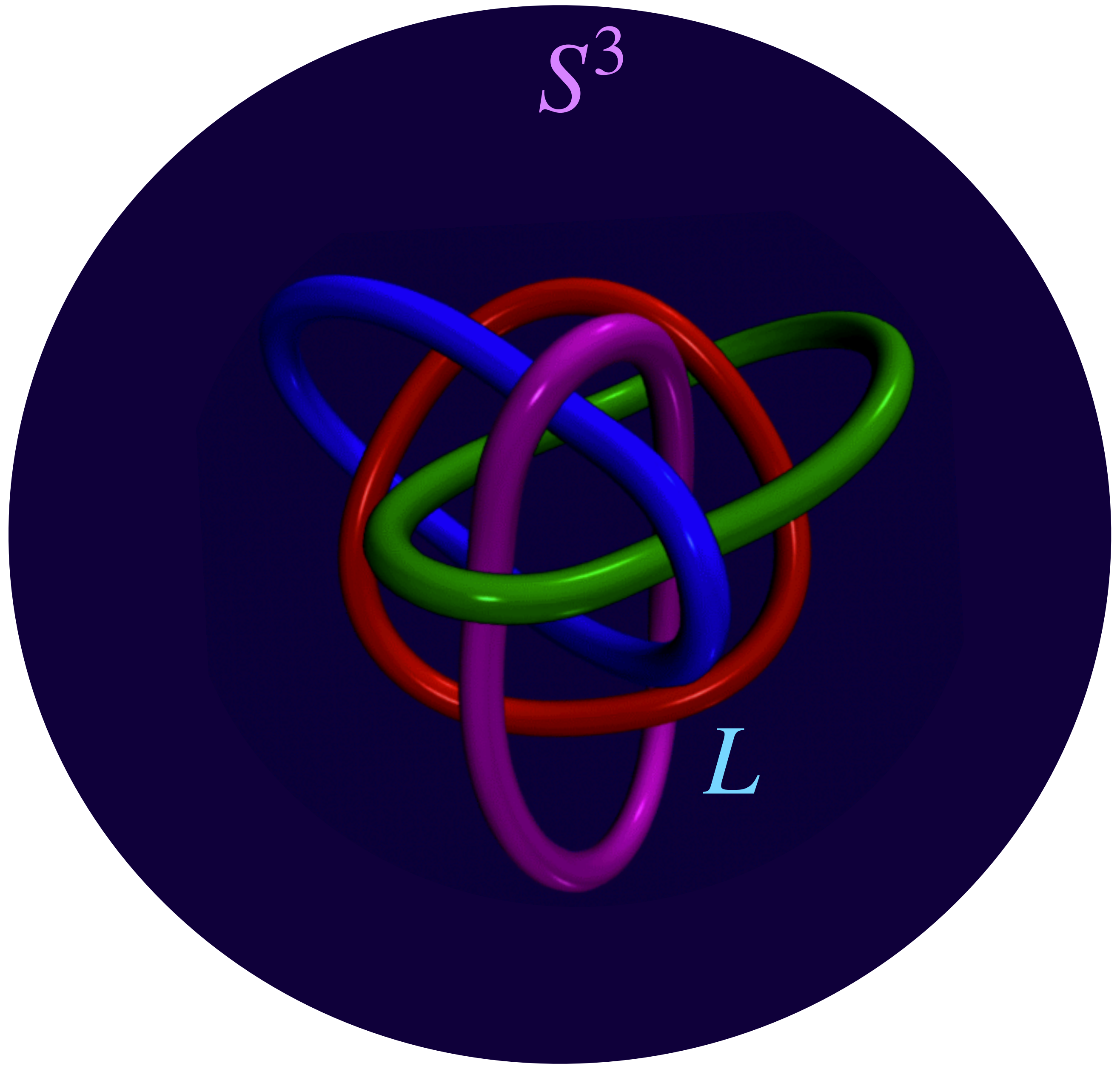

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Question: Which links are biorderable?

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↳ **Which braided links are biorderable?**

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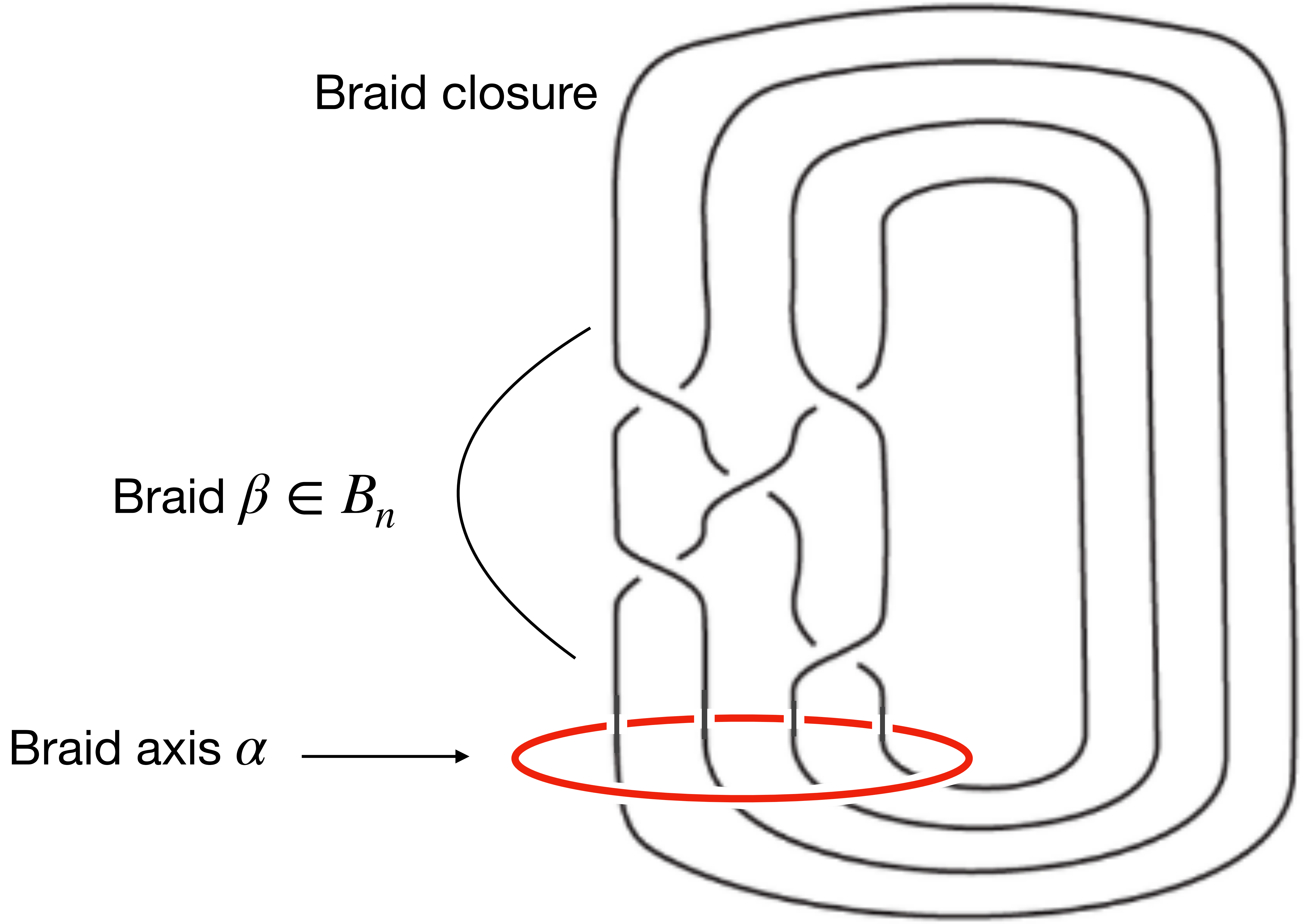

Braided link

$L = \beta \cup \alpha$

Quick Aside: What is the braid group?

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B_n is the group of braids on n strands

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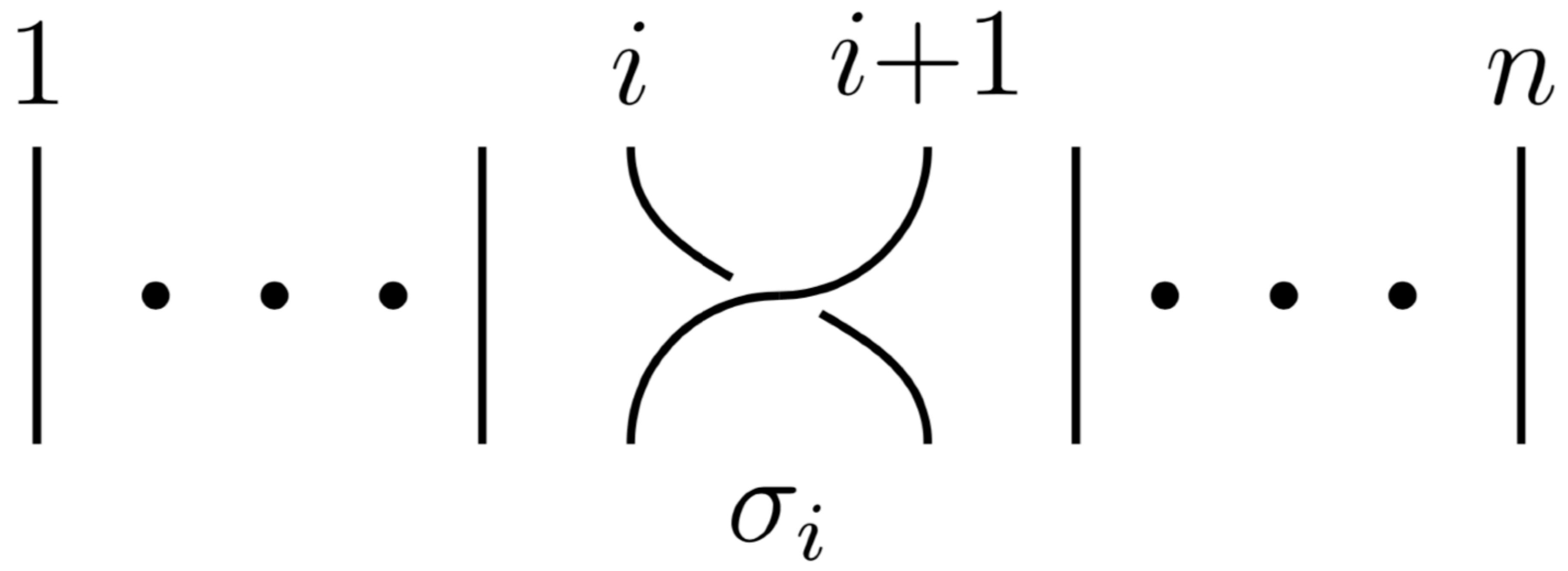
Generators



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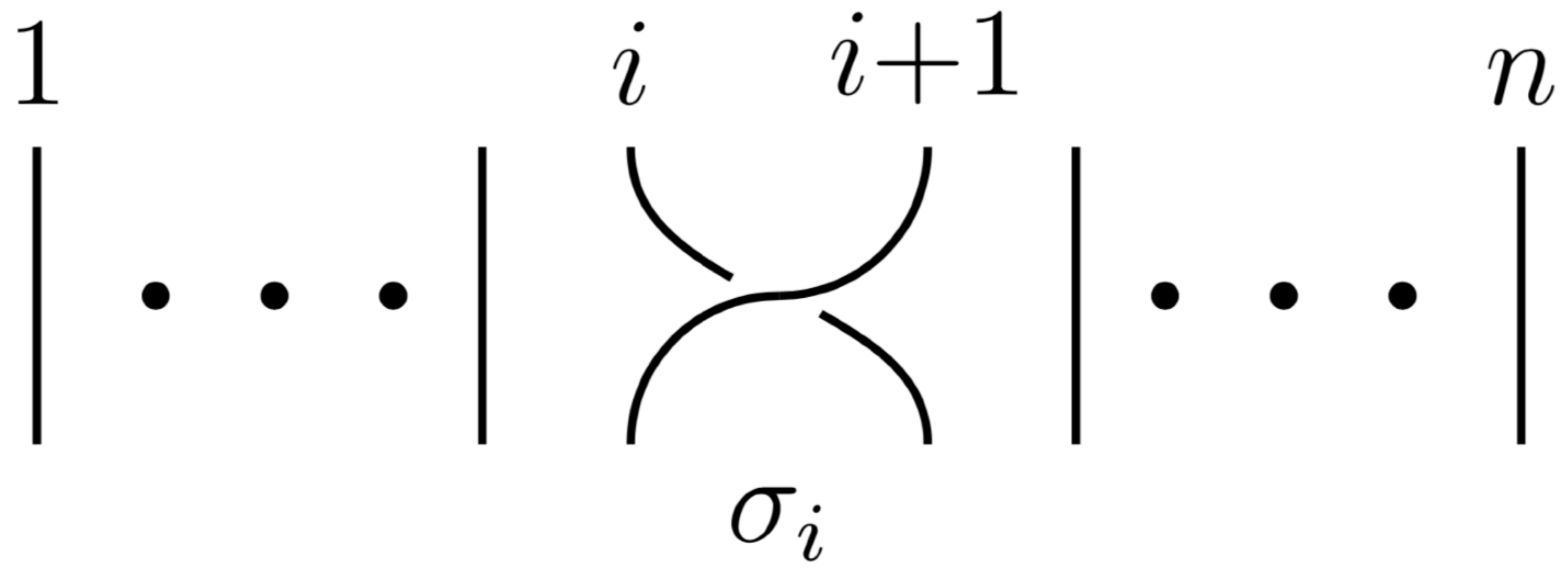
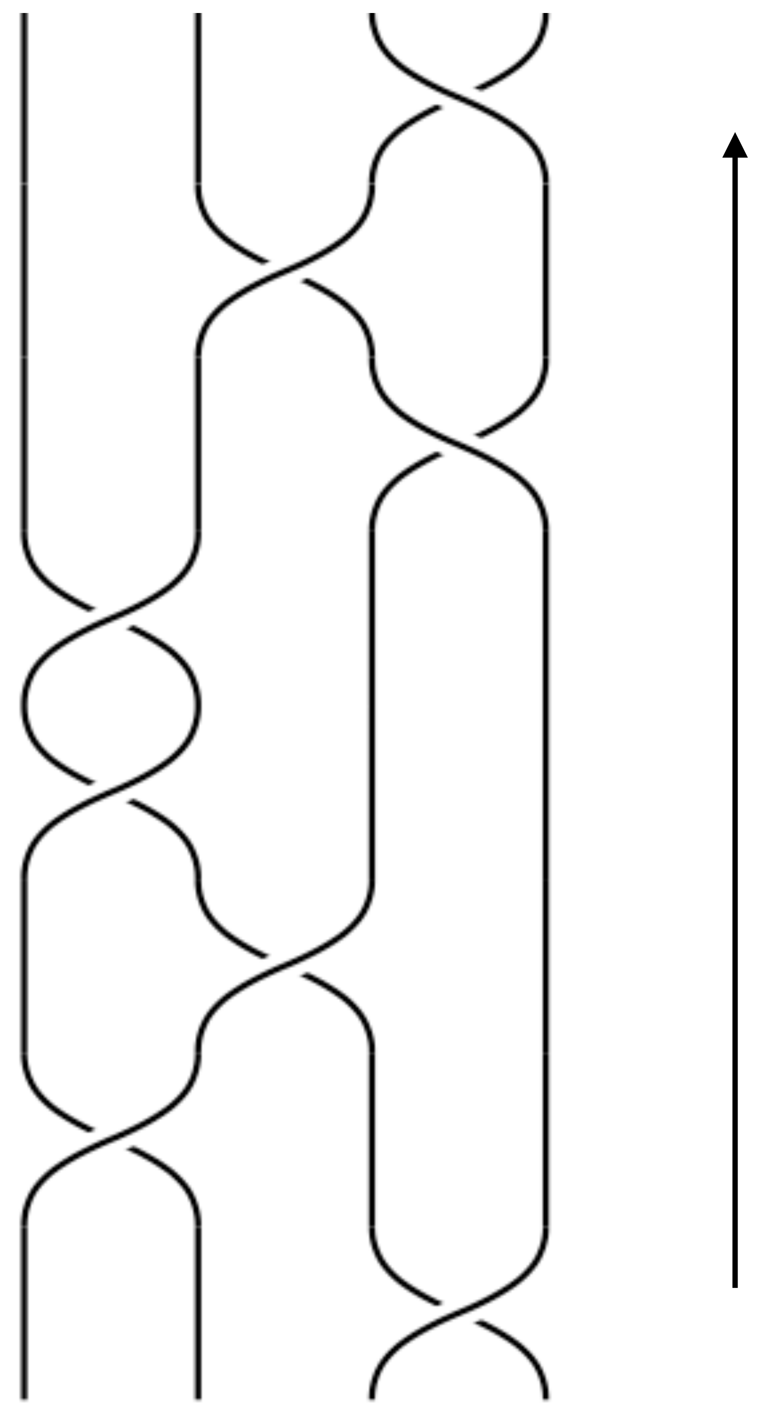


$$\sigma_3\sigma_1\sigma_2\sigma_1^2\sigma_3^{-1}\sigma_2\sigma_3^{-1} =$$



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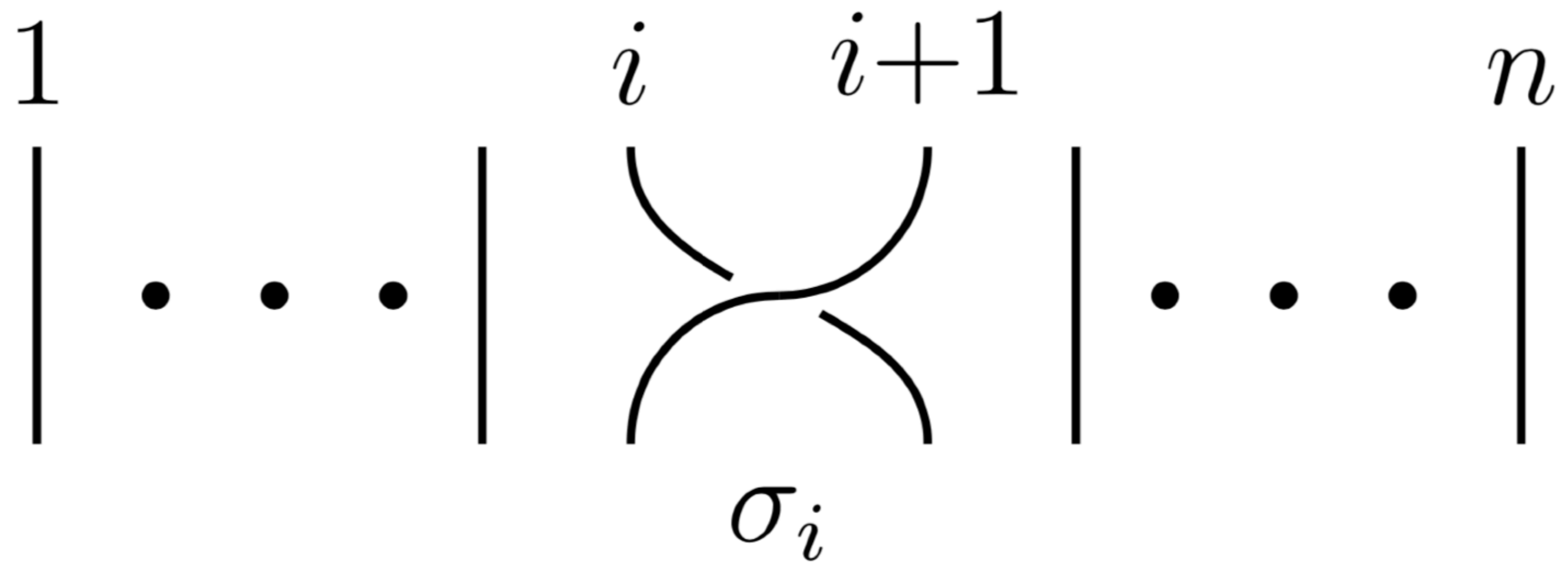
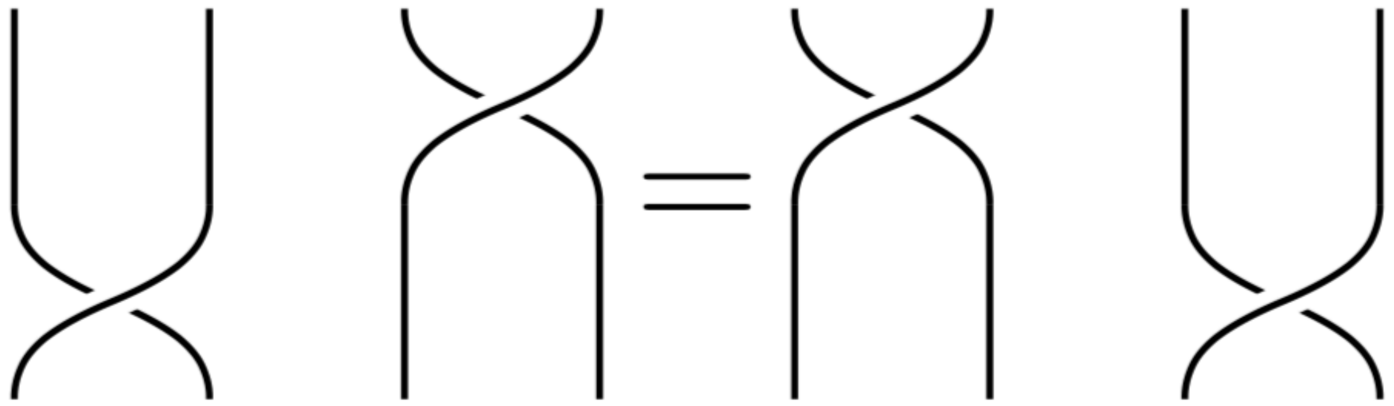


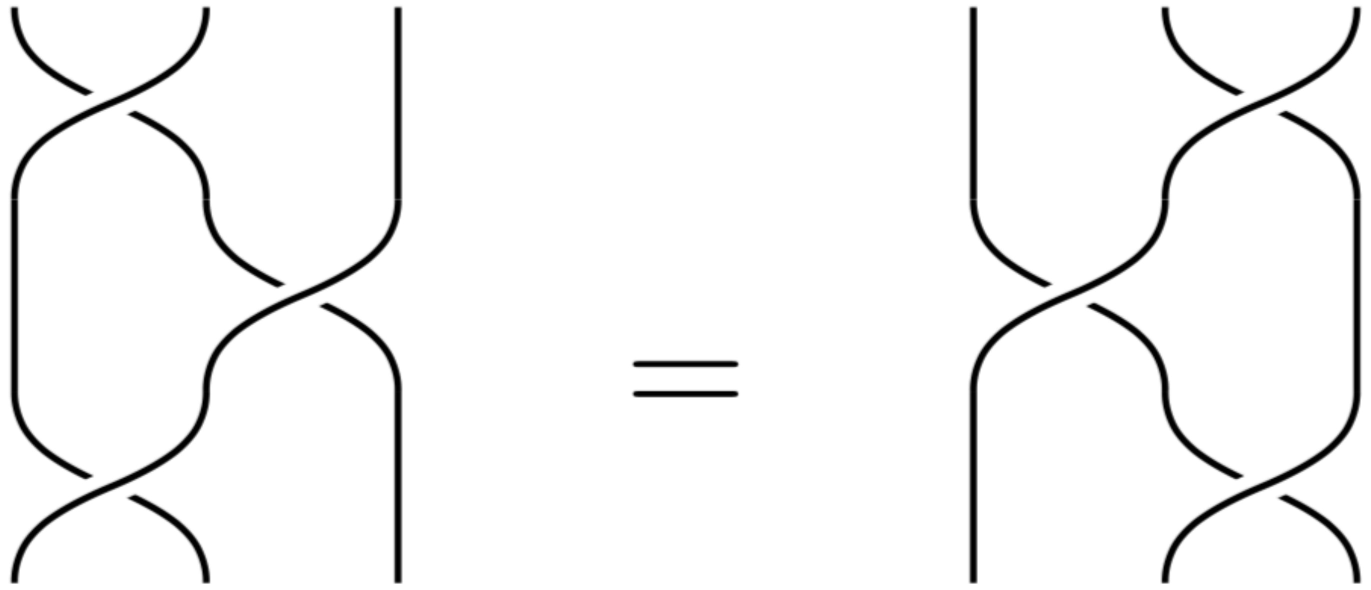
Far commutativity relation

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i-j| > 1$$

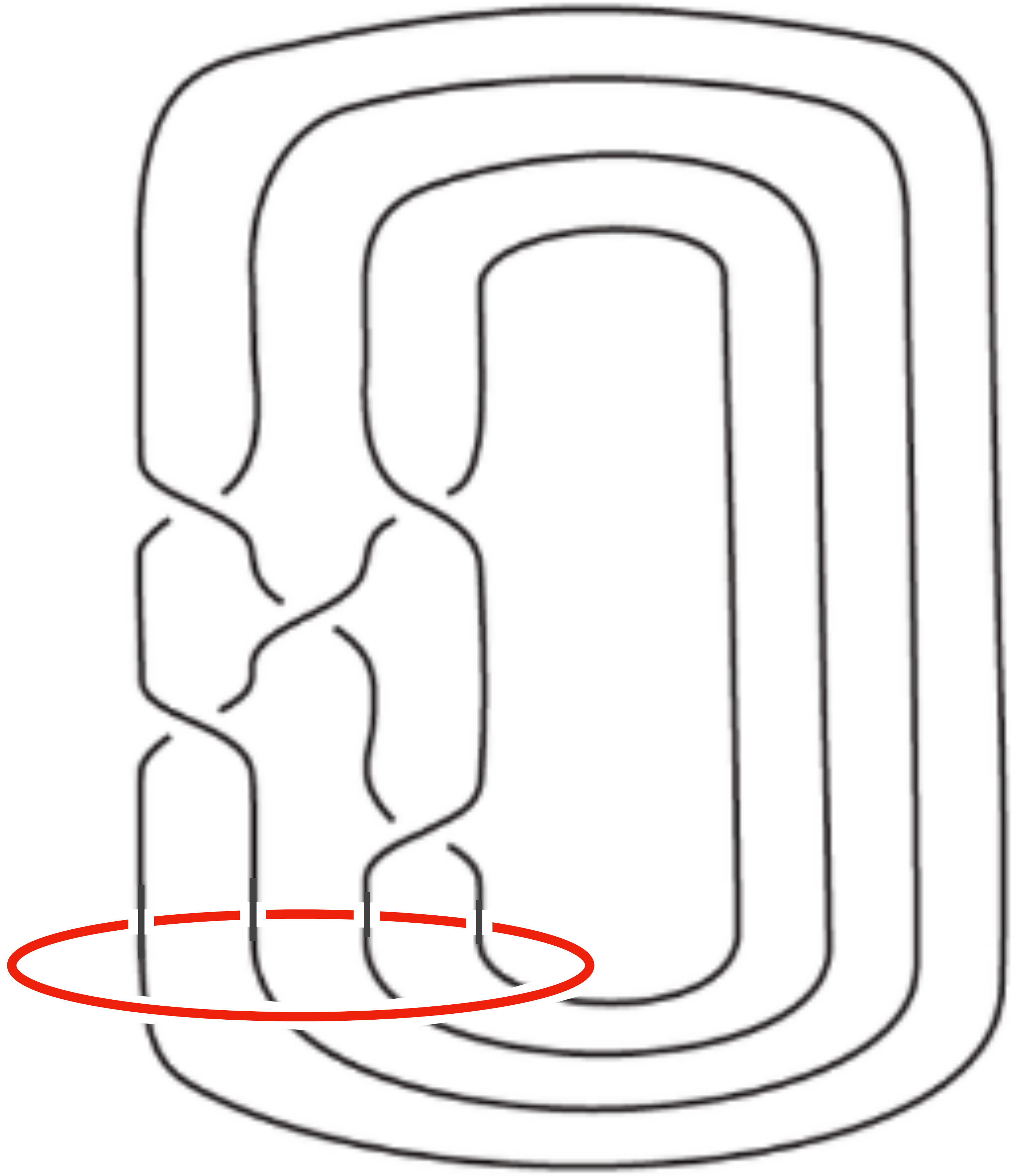


The braid relation

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$



Which braided links are biorderable?



Algorithm (Johnson-S.-Turner 2024)

If a braided link is NOT biorderable

then the algorithm returns "no" and a proof that the link is not biorderable.

If a braided link is biorderable

then the algorithm does not terminate.

Implemented in Python
Available on GitHub

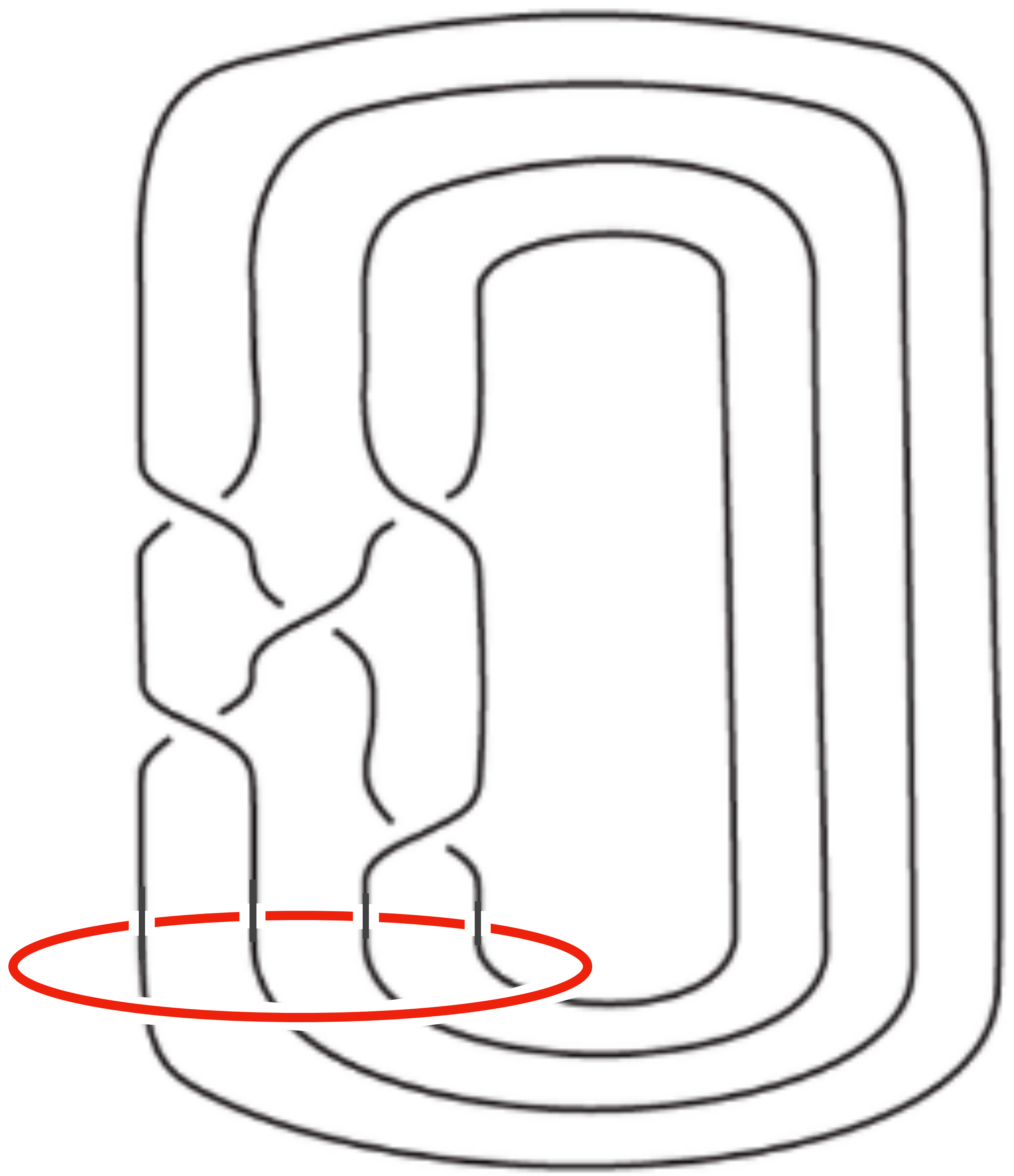
Theorem (Johnson-S.-Turner 2024)

The braided link $\sigma_1\sigma_2^{-3}$ is NOT order preserving.

The braided links $\sigma_1\sigma_2^{2k+1}$ are NOT order preserving.

Inspired by an algorithm of Calagari-Dunfield for left-orderable groups

Key Theorem for braided links



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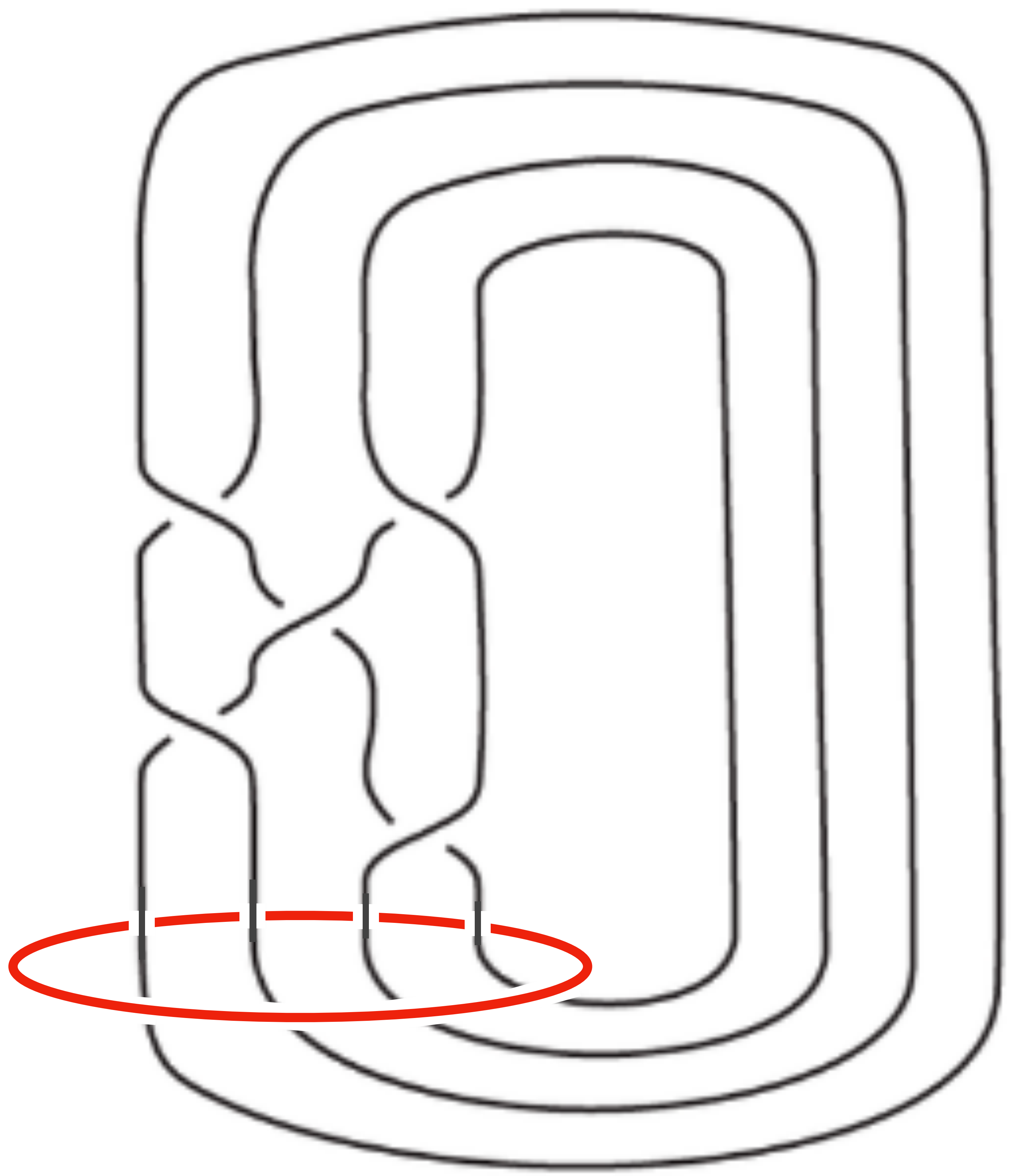

Braids act on the Free group

$$B_n \to Aut(F_n)$$

$$F_n = < x_1, x_2, \cdots, x_n >$$

$$\sigma_i \mapsto \begin{cases} x_i \mapsto x_{i+1} \\ x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1} \\ x_j \mapsto x_j \end{cases}$$

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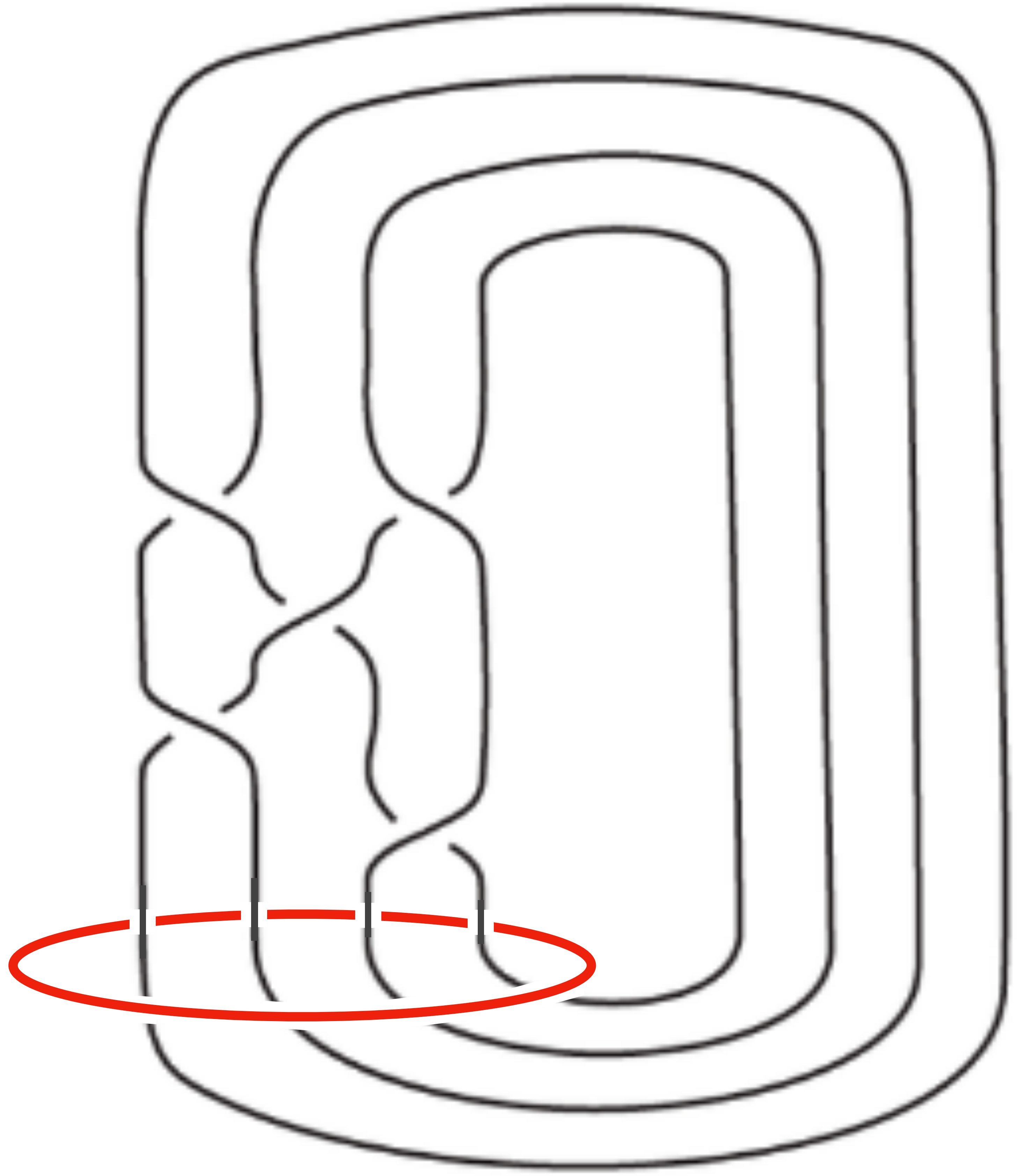

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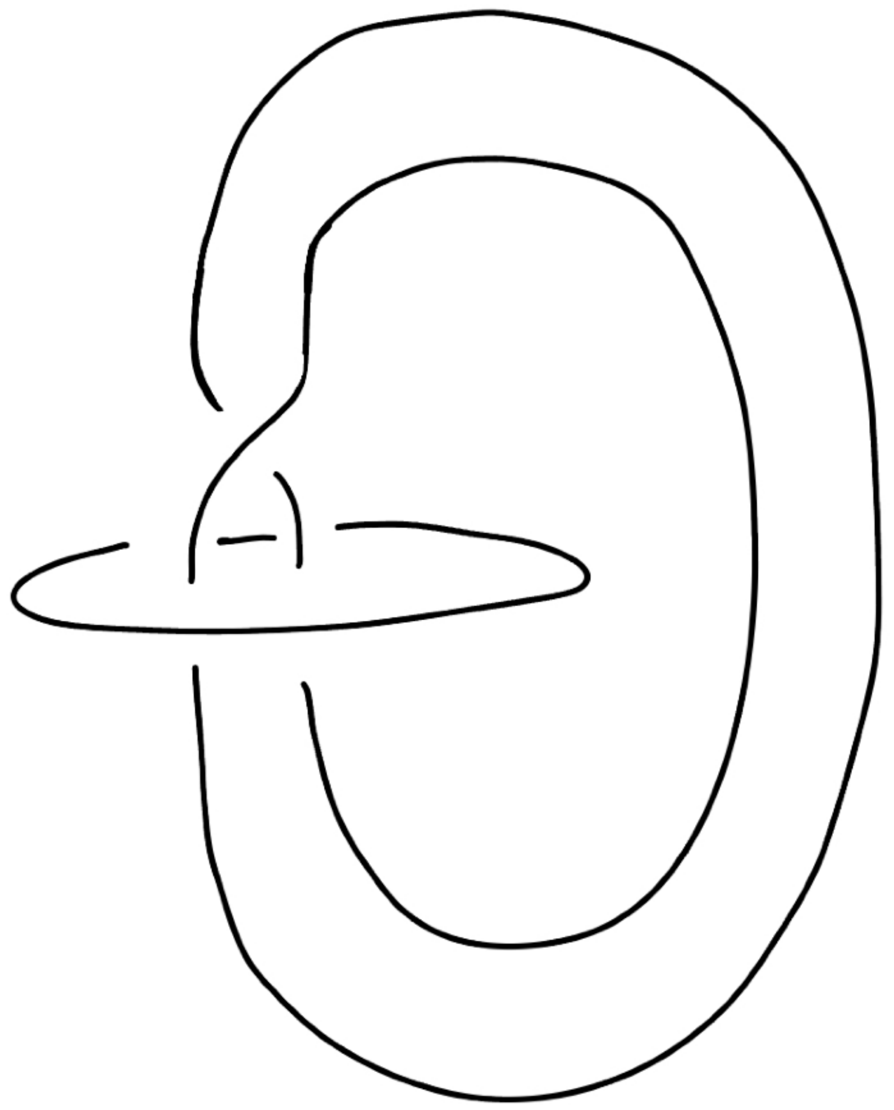
Kin-Rolfsen (2018)

A braided link is biorderable $\Longleftrightarrow$ There exists biordering of the free group that is invariant under the action of β.

i.e. if $x < y$ then $\beta(x) < \beta(y)$

Say “β is order preserving” or “OP”

Example: Is σ_1 order preserving?



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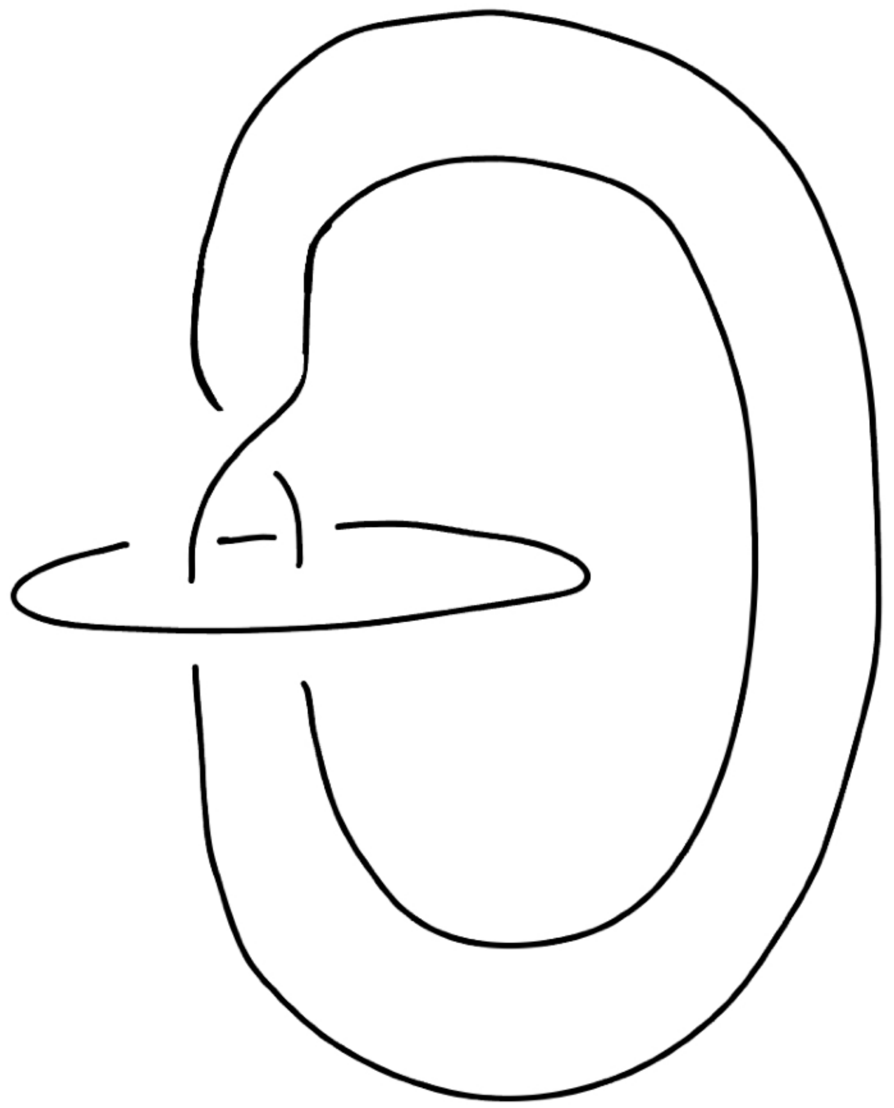

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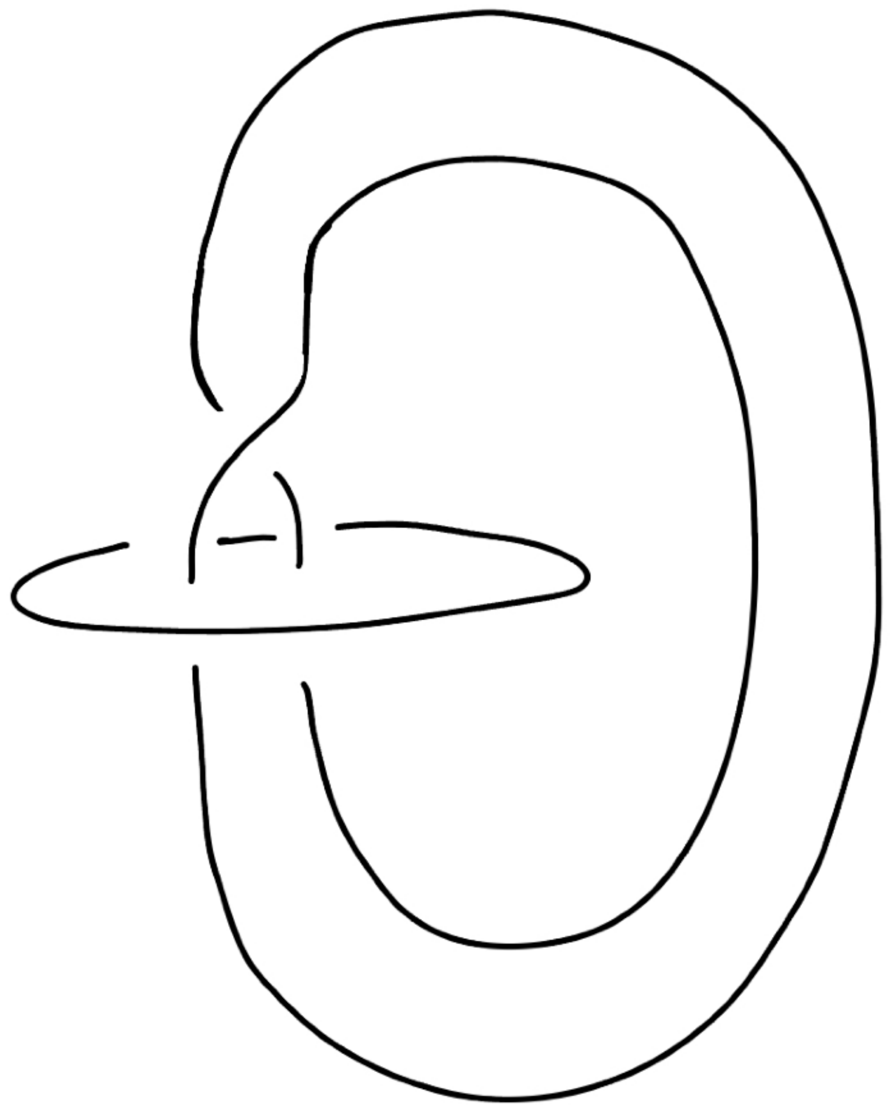

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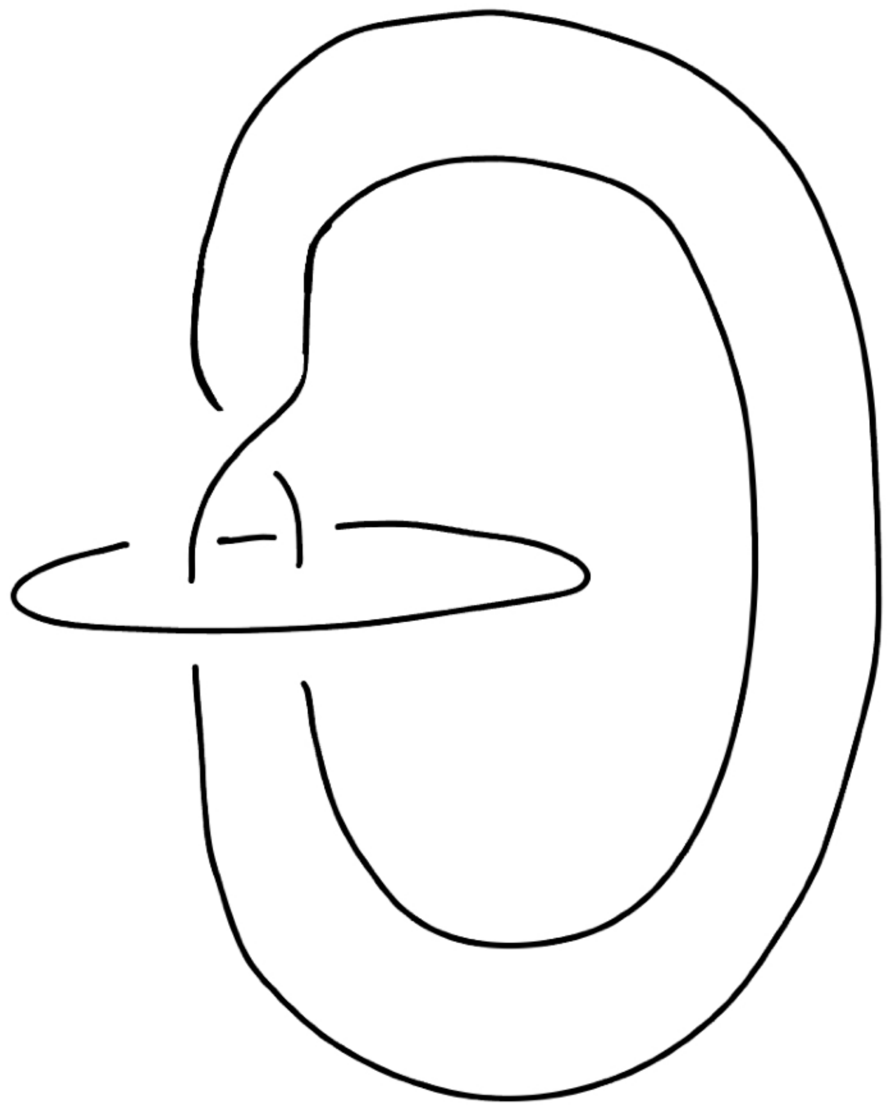

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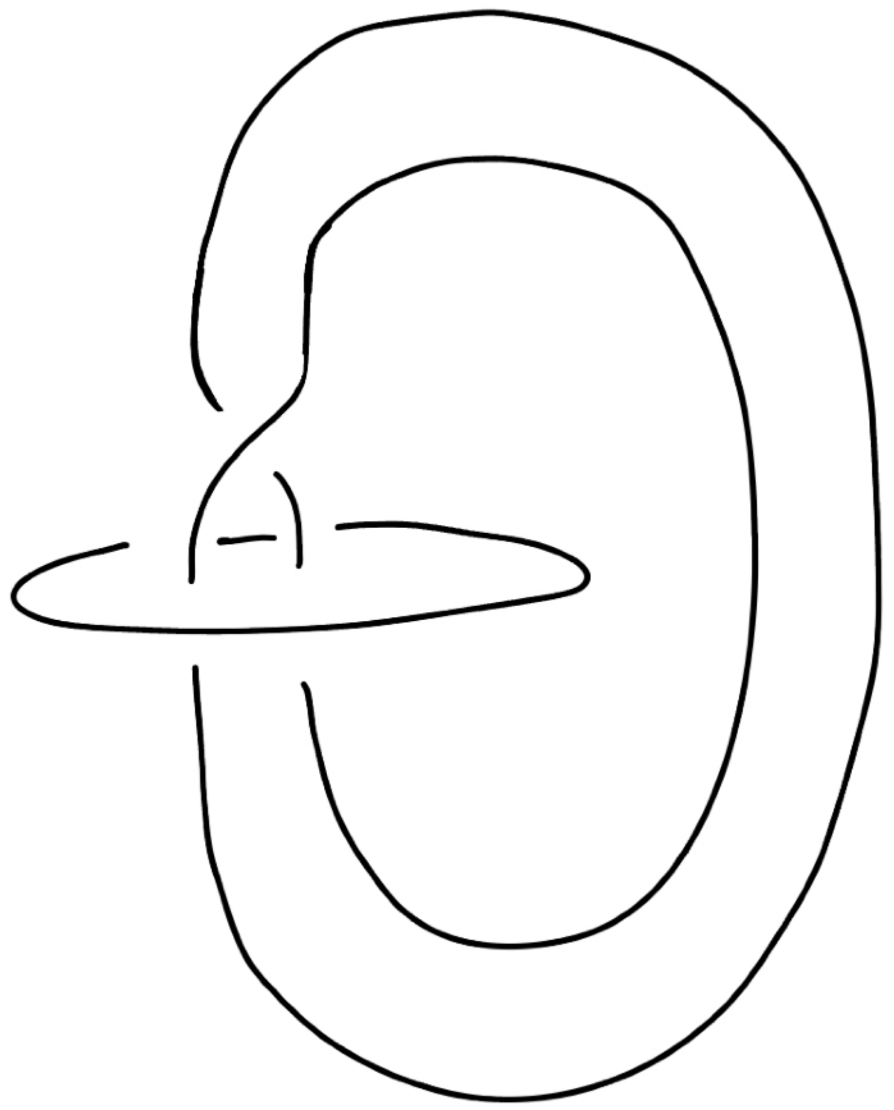

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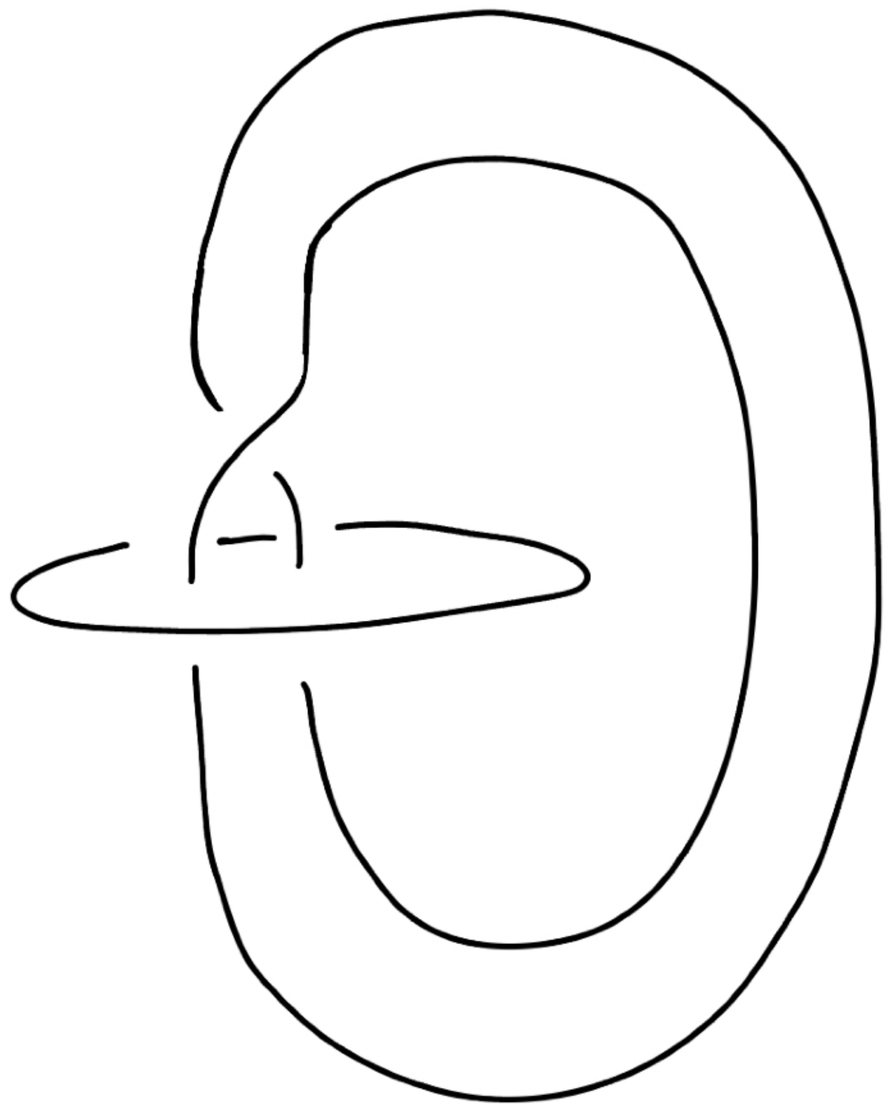

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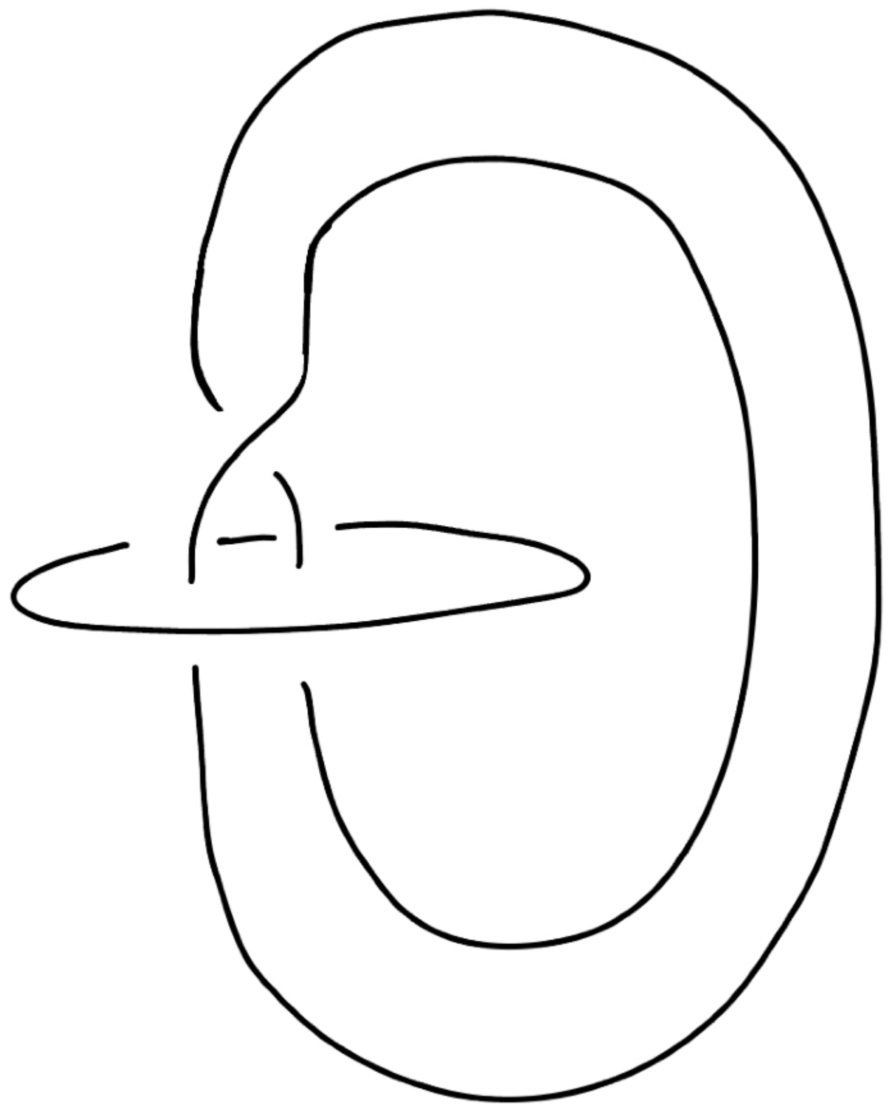

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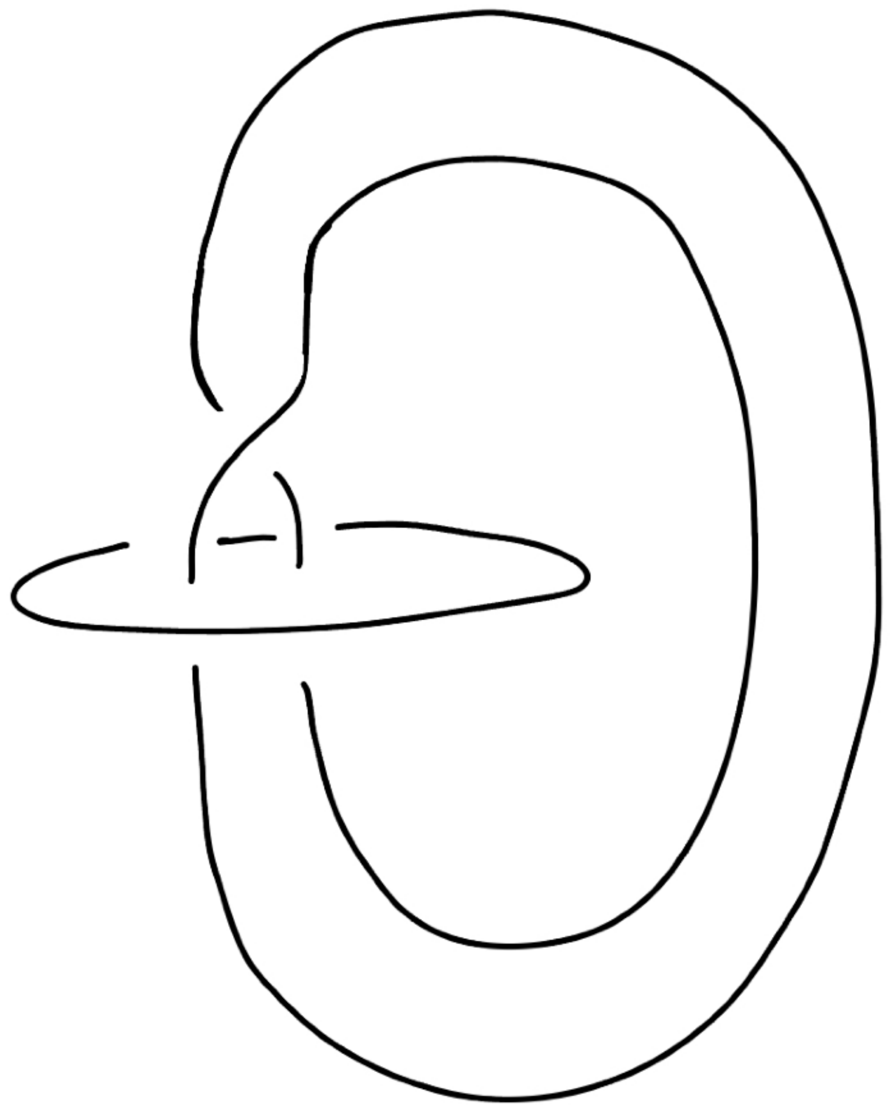

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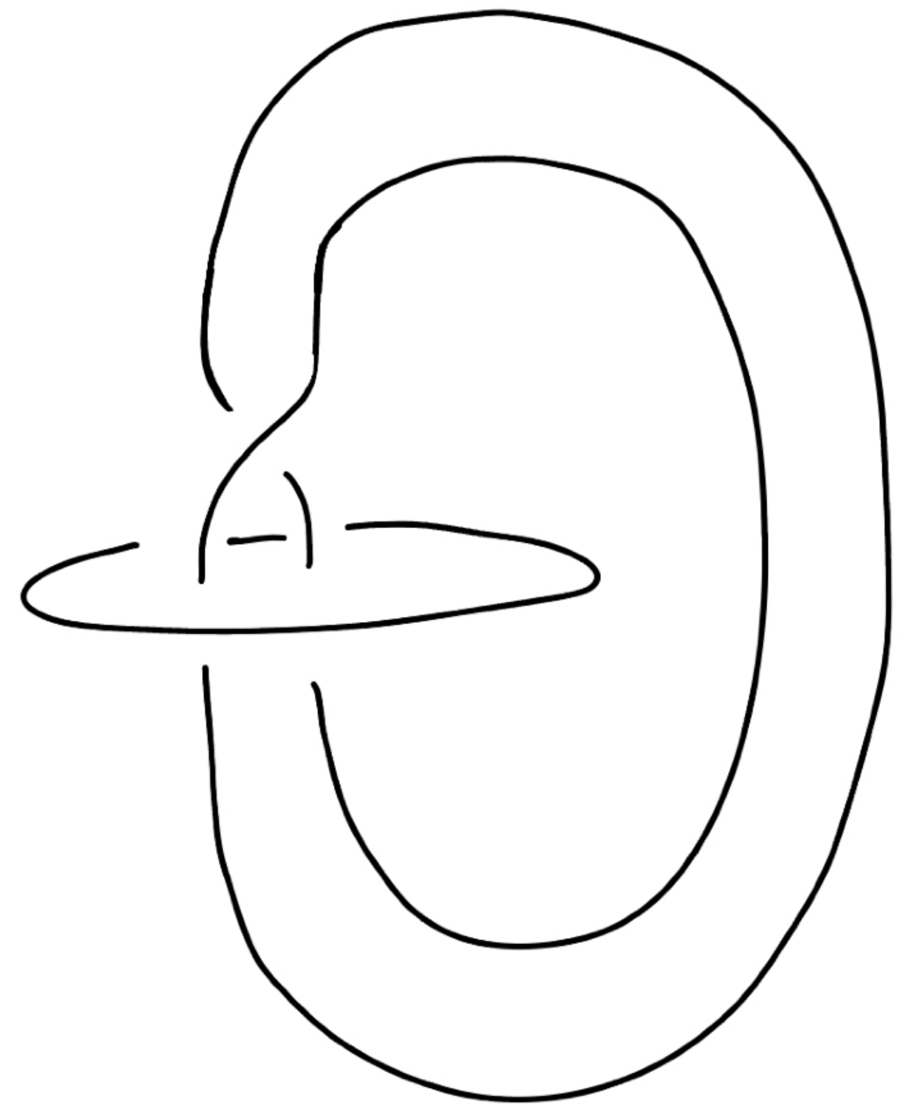

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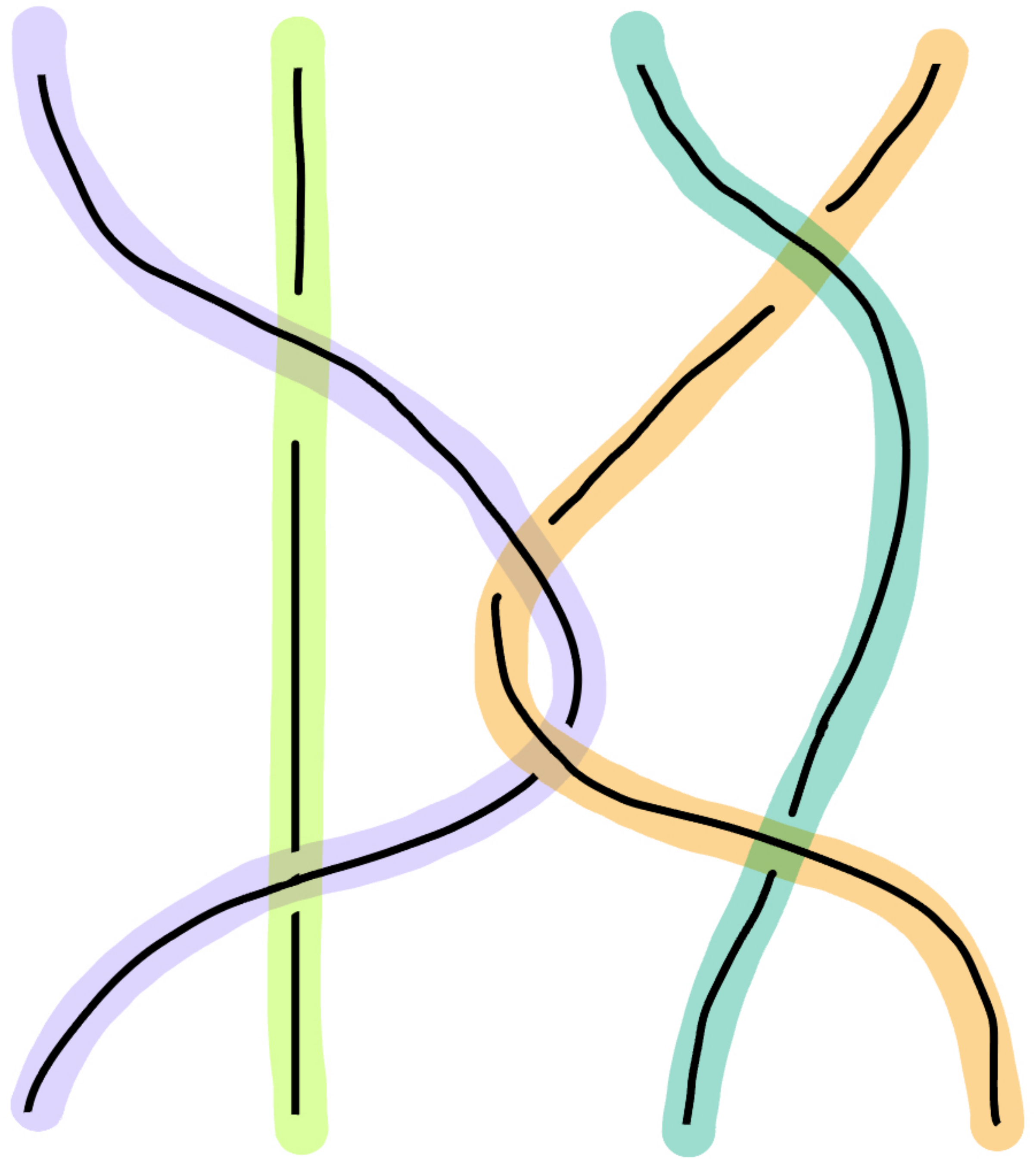
✗

$\Rightarrow$ For all i, σ_i is not OP.

Example: Pure braids are order preserving.



β is a pure braid $\Longleftrightarrow$ (**Kin-Rolfsen (2018)**) β preserves every *standard order* of F_n.

(Standard orders on F_n are defined using lower central series)

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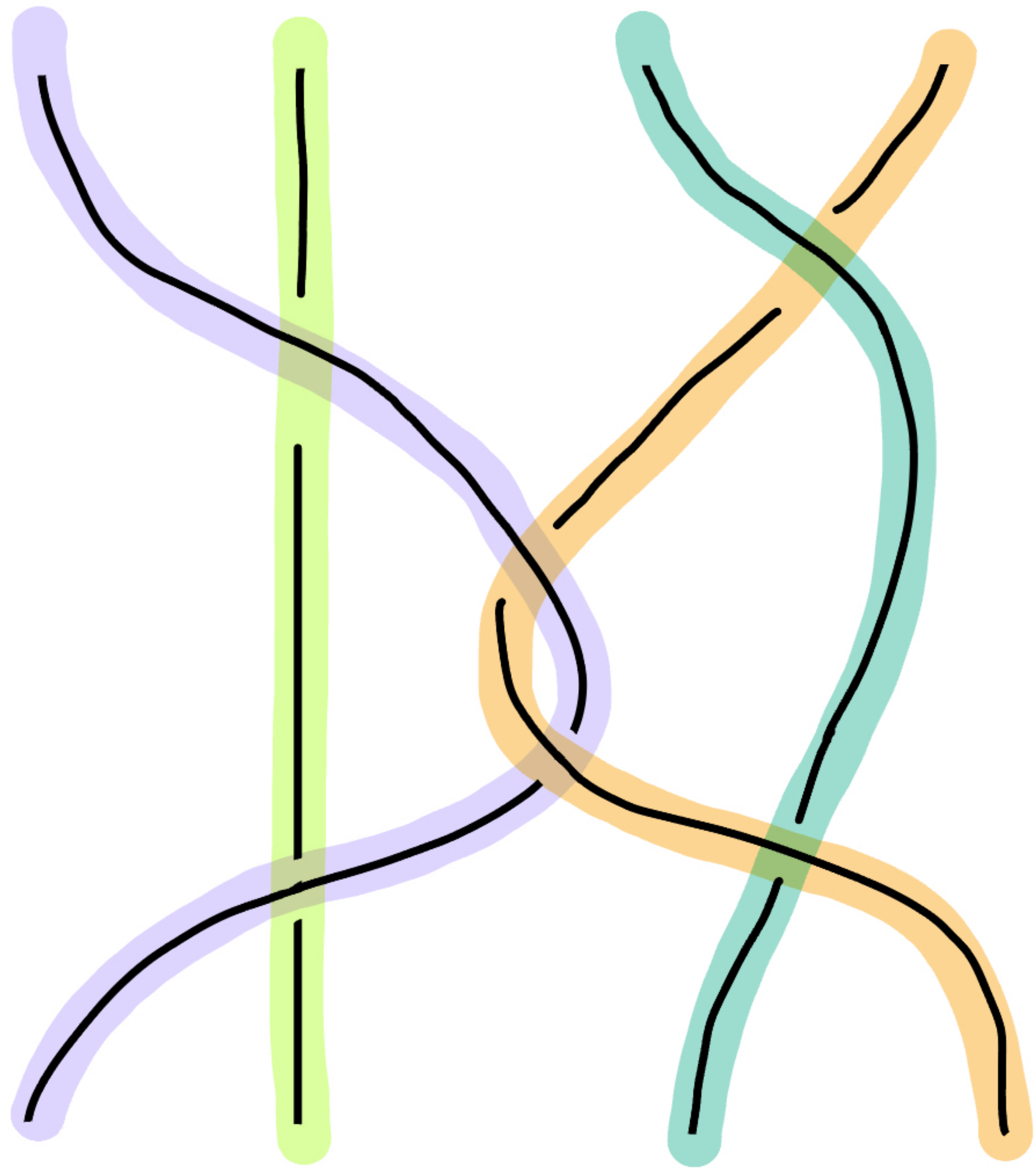

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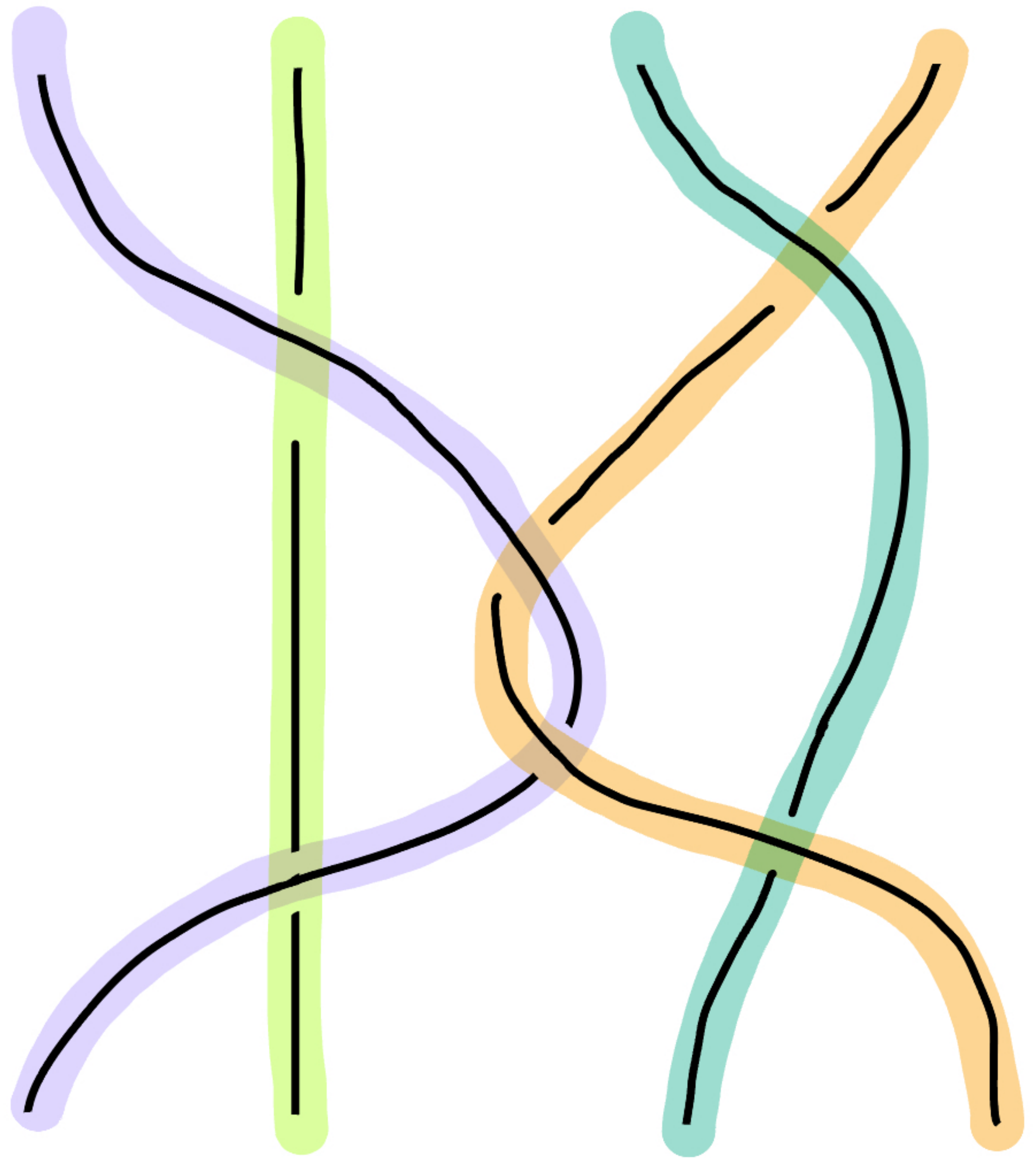

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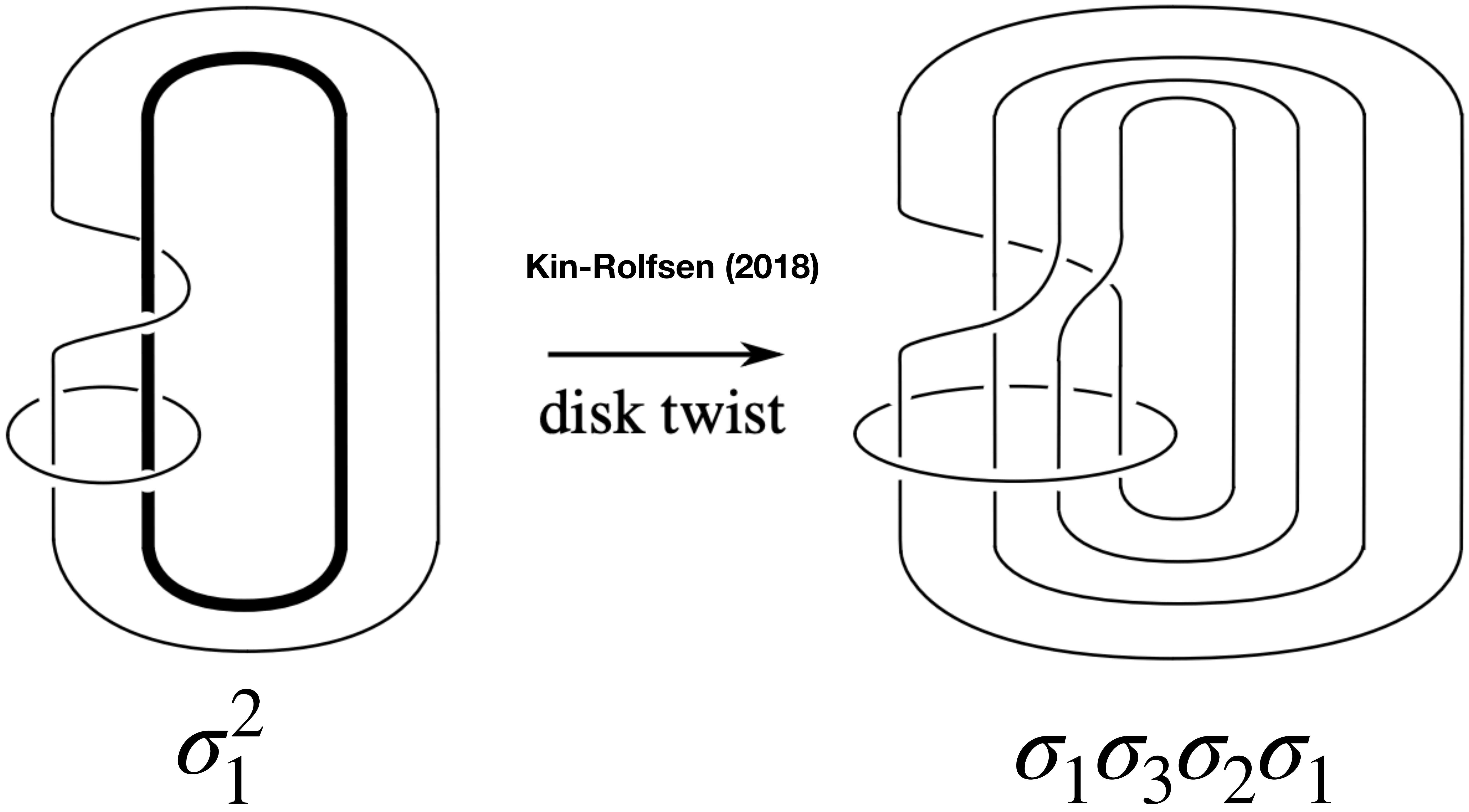
$\Rightarrow$ For every braid β, there is an integer k for which β^k is a pure braid, so β^k is OP.

Example: $\sigma_1\sigma_3\sigma_2\sigma_1$ is order preserving and not pure!



Fun Facts about order preserving braids

↳ **Why is it algebraically tricky to spot OP braids?**

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For $\alpha, \beta \in B_n$ both order preserving, $\alpha \cdot \beta$ may or may not be order preserving.

$\Longrightarrow$ The set of order preserving braids is not a subgroup!

How the algorithm works

- **Input a braid $\beta \in B_n$**
- **Attempt to build all biorders on F_n preserved by β, look for contradictions along the way.**

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$x < y$ iff $0 < x^{-1}y$ iff $x^{-1}y \in P$

Give a biorder on a group	$\Longleftrightarrow$	Give a positive cone of the group.

*preserved by β

If $x < y$ then $\beta(x) < \beta(y)$

A positive cone P of a group G is

A. $P \subset G$

B. $P \cdot P \subset G$

C. For all $x \in G$, $x^{-1}Px \in P$

D. For all $x \in G$, either $x \in P$ or $x^{-1} \in P$

*preserved by β, i.e. $\beta(P) \subset P$

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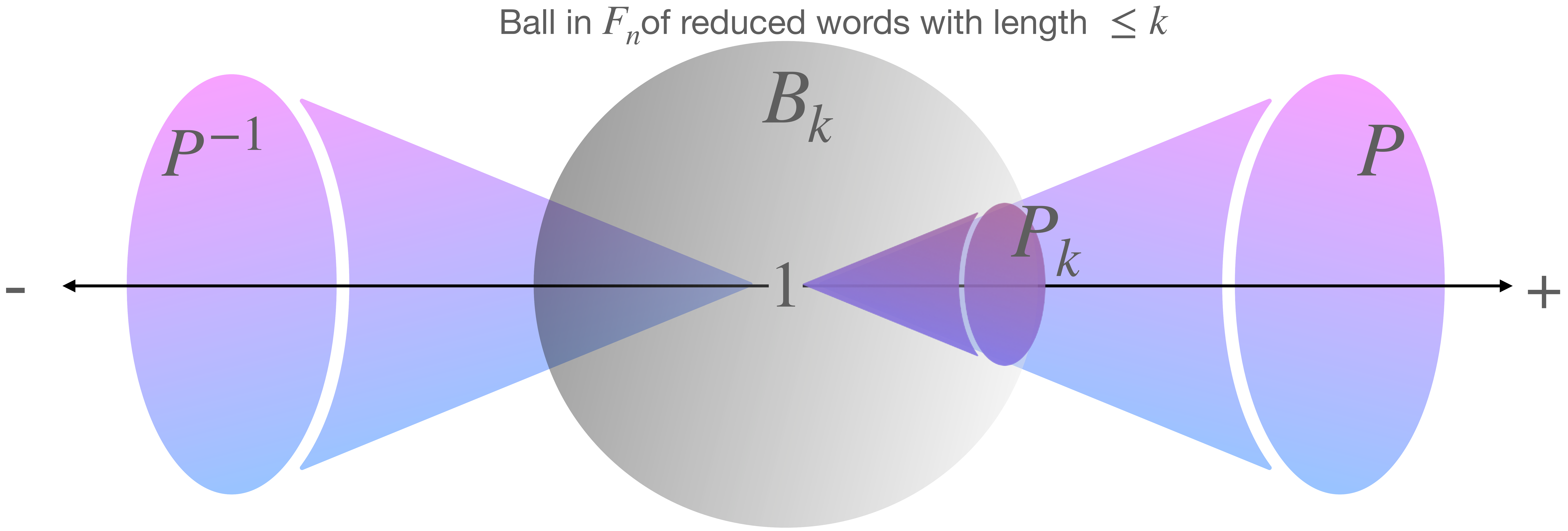

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