Applications of dimension theory to embeddability problems in topological data analysis: the case study of the Gromov-Hausdorff distance

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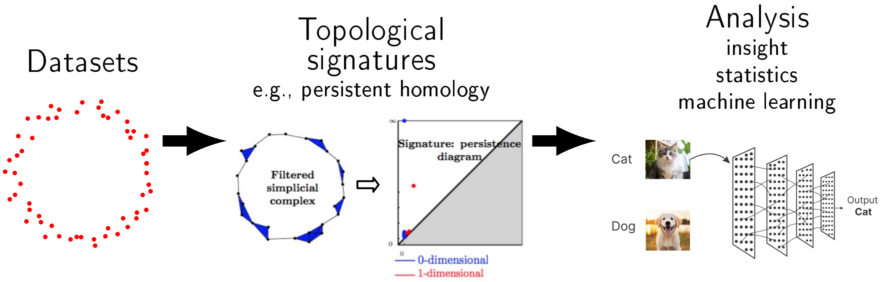
Outline

1. What is Topological Data Analysis (TDA)?
2. Why embeddability problems pop up in the applications?
3. Gromov-Hausdorff distance
4. Main results

N.Z., *Coarse and bi-Lipschitz embeddability of subspaces of the Gromov-Hausdorff space into Hilbert spaces.* `arXiv:2303.04730v2`.

What is **T**opological **D**ata **A**nalysis?

It defines and studies the computational aspects of topologically inspired invariants to analyse datasets.



Consider checking the youtube channel of the **A**pplied **A**lgebraic **T**opology **R**esearch **N**etwork.

TDA uses tools and motivates questions: algebraic topology, discrete mathematics, representation theory, **metric geometry**, **dimension theory**,...

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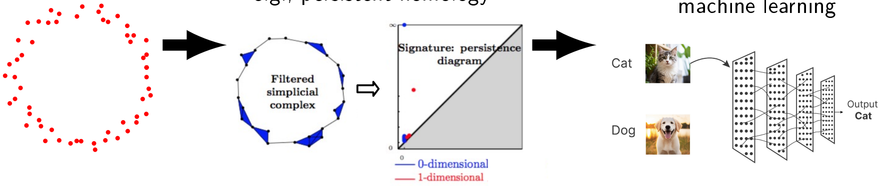
Datasets

Topological signatures
e.g., persistent homology

Analysis
insight
statistics
machine learning



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TDA uses tools and motivates questions: algebraic topology, discrete mathematics, representation theory, **metric geometry**, **dimension theory**,...

Datasets $\longrightarrow$ Topological signatures $\longrightarrow$ Analysis (machine learning).

To exploit 'topological signatures' in machine learning pipelines, we need to represent them in a Hilbert space (the existence of a scalar product is crucial).

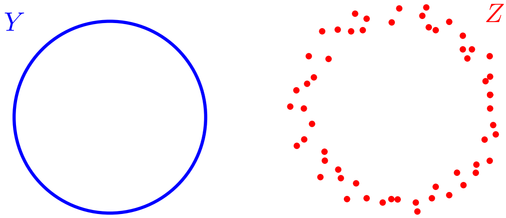
Consider a map

$$\phi\colon \mathcal{X} = \{\text{topological signatures}\} \to H, \text{ Hilbert space.}$$

- Is ϕ efficiently computable?
- Does ϕ provide a good representation of $\mathcal{X}$ into H?
  Is ϕ a 'reasonably good metric embedding' of $\mathcal{X}$ into H?

		are X and Y close in $\mathcal{X}$?	
		Yes	No
are $\phi(X)$ and $\phi(Y)$ close in H?	Yes	true positive	false positive
	No	false negative	true negative

The Gromov-Hausdorff distance d_{GH} is a distance between compact metric spaces. It estimates how far two spaces are from being isometric. Successfully deployed in TDA as a theoretical framework for shape and dataset comparison (F. Mémoli, 2007).



Datasets metric spaces ⟶ Topological signatures ⟶ Analysis (machine learning)

↑ how well does a topological signature reflect the difference?

Persistence diagrams. F. Chazal, D. Cohen-Steiner, L.J. Guibas, F. Mémoli, S.Y. Oudot (2009); F. Chazal, V. De Silva, S.Y. Oudot (2014):

$$d(\mathrm{Dgm}(X), \mathrm{Dgm}(Y)) \leq 2 d_{GH}(X, Y).$$

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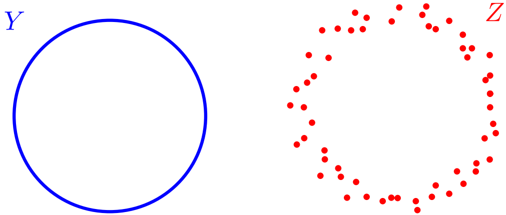

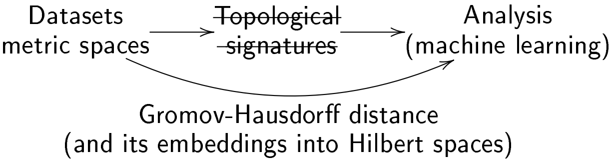

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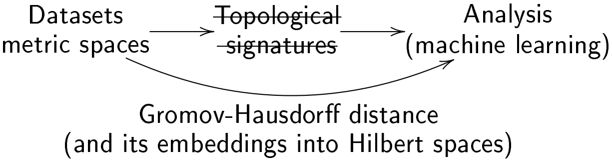
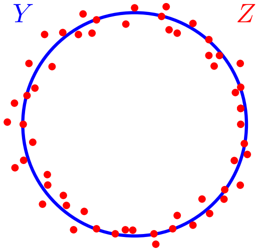
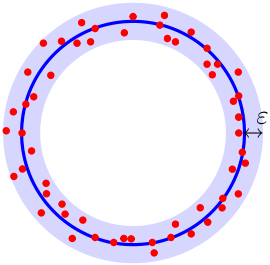


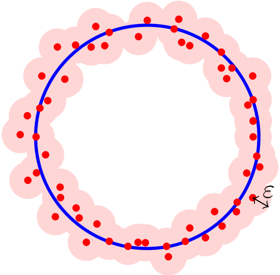
Computing the Gromov-Hausdorff distance between finite metric spaces is NP-hard (even approximating it within a factor of 3 for trees with unit edge length; P.K. Agarwal, K. Fox, A. Nath, A. Sidiropoulos, Y. Wang, 2018).

Question

Can the Gromov-Hausdorff space be 'reasonably well embedded' into a (finite-dimensional) Hilbert space?

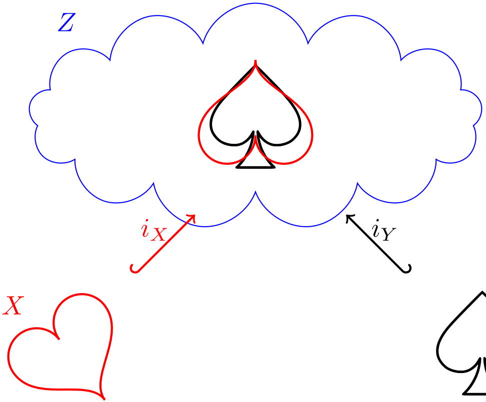






If Y and Z are two subsets of a metric space (X, d), their Hausdorff distance is

$$d_H(Y, Z) = \inf\{\varepsilon > 0 \mid Y \subseteq B_\varepsilon(Z) \text{ and } Z \subseteq B_\varepsilon(Y)\}.$$



The Gromov-Hausdorff distance (M. Gromov 1981, D.A. Edwards 1975) between two metric spaces X and Y is

$$d_{GH}(X,Y) = \inf_{Z \text{ metric space}} \inf_{\substack{i_X : X \hookrightarrow Z \text{ isometric embedding} \\ i_Y : Y \hookrightarrow Z \text{ isometric embedding}}} d_H(i_X(X), i_Y(Y)).$$

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If X and Y are compact, then

- $d_{GH}(X,Y) < \infty$,
- $d_{GH}(X,Y) = 0 \Leftrightarrow X \overset{\text{isom}}{\cong} Y$, and
- d_{GH} satisfies the triangular inequality.

The Gromov-Hausdorff space is the metric space

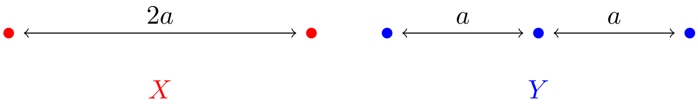
$$\mathcal{GH} = (\{[X]_{\text{isom}} \mid X \text{ compact metric space}\}, d_{GH}).$$

Diameter: $\operatorname{diam}\colon \mathcal{GH} \to \mathbb{R}_{\geq 0}$.

The distance between the images of two metric spaces can be upper bounded:

$$|\operatorname{diam} X - \operatorname{diam} Y| \leq 2 d_{GH}(X, Y);$$

but it cannot be lower bounded:



Distance set: $\mathcal{D}(X) = \{d_X(x, x') \mid x, x' \in X\} \subseteq \mathbb{R}$.

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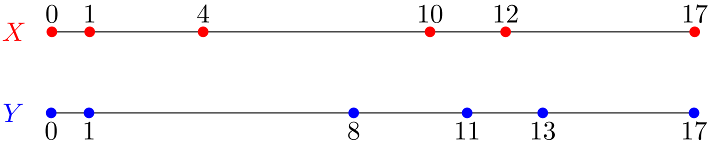
but it cannot be lower bounded (G. S. Bloom, 1977):



$$\mathcal{D}(X) = \mathcal{D}(Y) = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17\},$$

but X and Y are not isometric.

Question

Can the Gromov-Hausdorff space be 'reasonably well embedded' (bi-Lipschitz or coarsely embedded) into a (finite-dimensional) Hilbert space?

Non-existence. Two strategies to show it:

1. using notions of dimension;
2. finding in X 'weird' (not embeddable) subspaces.

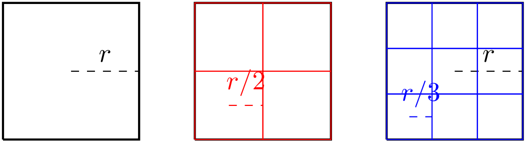
1. **Strategy using dimension.**
   - Let $\dim$ be a 'dimension' such that, if X can be embedded into Y, then $\dim X \leq \dim Y$.
   - If $\dim X = n$ (possibly, $n = \infty$), then X cannot be embedded into any Y with $\dim Y < n$.

Definition

A map $\phi\colon X \to Y$ between metric space is a bi-Lipschitz embedding if there is $L > 0$ such that, for every $x, y \in X$,

$$L^{-1} \cdot d_X(x,y) \leq d_Y(\phi(x), \phi(y)) \leq L \cdot d_X(x,y).$$

We use the Assouad dimension $\dim_A$ (A. Assouad, 1983; M.G. Bouligand, 1928).



A ball of radius r can be covered by $2^2 = \left(\frac{r}{r/2}\right)^{\mathbf{2}}$ balls of radius $r/2$ and $3^2 = \left(\frac{r}{r/3}\right)^{\mathbf{2}}$ balls of radius $r/3$.

- If $\phi\colon X \to Y$ is a bi-Lipschitz embedding, then $\dim_A X \leq \dim_A Y$.
- $\dim_A \mathbb{R}^n = n$.
- Hence, if $\dim_A X = \infty$, X cannot be bi-Lipschitz embedded into any $\mathbb{R}^n$.

Theorem (N.Z.)

The family of (isometry classes) of finite subsets of $[0,1]$ endowed with the Gromov-Hausdorff distance has infinite $\dim_A$. Hence, it cannot be bi-Lipschitz embedded into any $\mathbb{R}^n$. The same holds for $\mathcal{GH}$.

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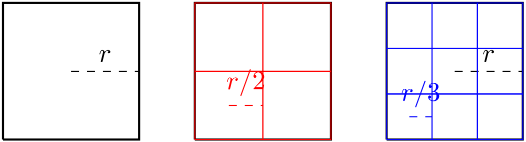

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A map $\phi\colon X \to Y$ between metric spaces is a coarse embedding if there are $\rho_-, \rho_+ \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\rho_- \to \infty$ and, for every $x, x' \in X$,

$$\rho_-(d_X(x,x')) \leq d_Y(\phi(x),\phi(x')) \leq \rho_+(d_X(x,x')).$$

We can use the asymptotic dimension asdim (M. Gromov, 1993), introduced as the large-scale counterpart of Lebesgue's covering dimension.

Let X be a metric space and $n \in \mathbb{N}$. The asymptotic dimension of X is at most n ($\operatorname{asdim} X \leq n$) if, for every $r \geq 0$, there is a uniformly bounded cover $\mathcal{U} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$ (i.e., $\sup_{U \in \mathcal{U}} \operatorname{diam} U < \infty$) of X such that, for every $i = 0, \dots, n$ and $U, V \in \mathcal{U}_i$, $B(U,r) \cap V \neq \emptyset$ if and only if $U = V$.

- If $\phi\colon X \to Y$ is a coarse embedding, then $\operatorname{asdim} X \leq \operatorname{asdim} Y$.
- $\operatorname{asdim} \mathbb{R}^n = n$.
- If $\operatorname{asdim} X = m$ (possibly, $m = \infty$), X cannot be coarsely embedded into any $\mathbb{R}^n$ with $n < m$.

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Let $\mathcal{GH}_n$ be the family of isometry classes of metric spaces with at most n points.

A symmetric matrix $n \times n$ has

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$

'degrees of freedom'.

d	1	2	3	$\cdot$	$\cdot$	$\cdot$	n
1	0	$*$	$*$	$*$	$*$	$*$	$*$
2		0	$*$	$*$	$*$	$*$	$*$
3			0	$*$	$*$	$*$	$*$
$\cdot$				0	$*$	$*$	$*$
$\cdot$					0	$*$	$*$
$\cdot$						0	$*$
n							0

Theorem (S. Iliadis, A.O. Ivanov, A.A. Tuzhilin, 2017)

$\mathcal{GH}_n$ contains subsets isometric to balls of $\mathbb{R}^{n(n-1)/2}$ of arbitrary radius.

Corollary

$\operatorname{asdim} \mathcal{GH}_n \geq n(n-1)/2.$

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Corollary

$\mathcal{GH}_n$ cannot be coarsely embedded into any $\mathbb{R}^m$ with $m < n(n-1)/2$.

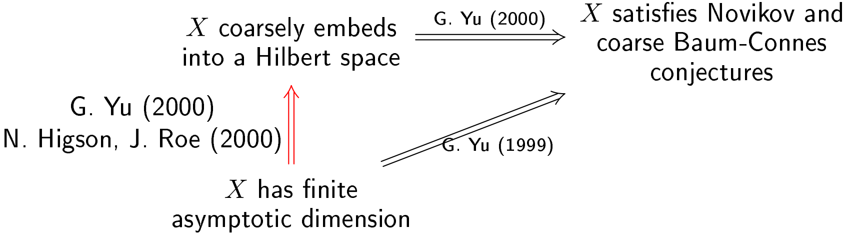
Since $\mathcal{GH} \supseteq \mathcal{GH}_\infty = \bigcup_n \mathcal{GH}_n$, we have the following result.

Corollary

$\operatorname{asdim} \mathcal{GH}_\infty = \infty$, and so $\operatorname{asdim} \mathcal{GH} = \infty$.
Hence, they cannot be coarsely embedded into any $\mathbb{R}^m$.

Theorem (N.Z.)

The family $\mathcal{GH}_n$ of isometry classes of metric spaces of at most n points has asymptotic dimension $n(n-1)/2$.



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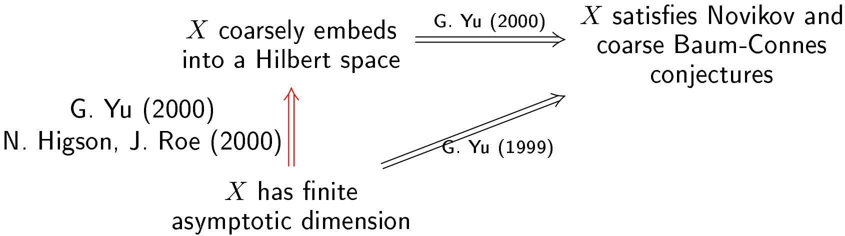

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Non-existence (false positives are unavoidable). Two strategies to show it:

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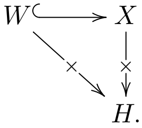

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Theorem (N.Z.)

The family of finite subsets of $\mathbb{R}$ endowed with d_{GH} cannot be coarsely embedded into any Hilbert space. In particular, this holds for $\mathcal{GH}_\infty$ and $\mathcal{GH}$.

It relies on two main ingredients.

1. The Euclidean-Hausdorff distance d_{EH}, a modification of d_{GH} for subsets of $\mathbb{R}^n$. On finite subsets of $\mathbb{R}$, d_{GH} and d_{EH} are bi-Lipschitz equivalent (S. Majhi, J. Vitter, C. Wenk, 2024).
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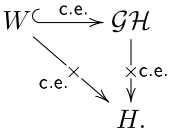

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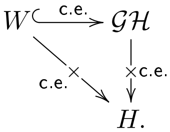

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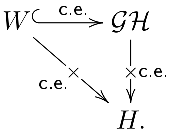

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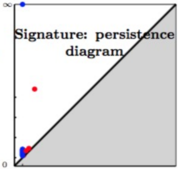
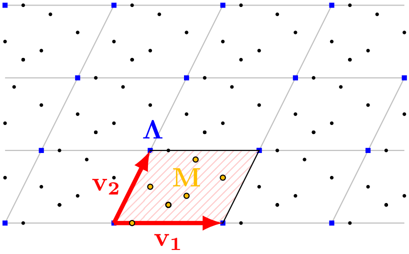
These notions and ideas have been applied in TDA:

Spaces of persistence diagrams.
M. Carriére, U. Bauer (2019); P. Bubenik, A. Wagner (2020), A. Wagner (2021); A. Mitra, Ž. Virk (2021, 2024); D. Bate, A.L. García Pulido (2023).



Spaces of periodic point sets.
A. Garber, Ž. Virk, N.Z. (2023).



Thank you very much for the attention.