What are Construction Schemes?

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Assume we want to build a structure of size ω_1

How can we proceed?

The most common approach is to build the structure using countable approximations

One might wonder...

Can we build an uncountable structure using finite approximations?

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Yes!!!!

This can be achieved using construction and capturing schemes, which were introduced by Todorcevic and are a generalization of the morasses of Velleman

Some of

The point of this technique is that many structural properties of the desired uncountable structure reduce to problems in amalgamating its finite substructures

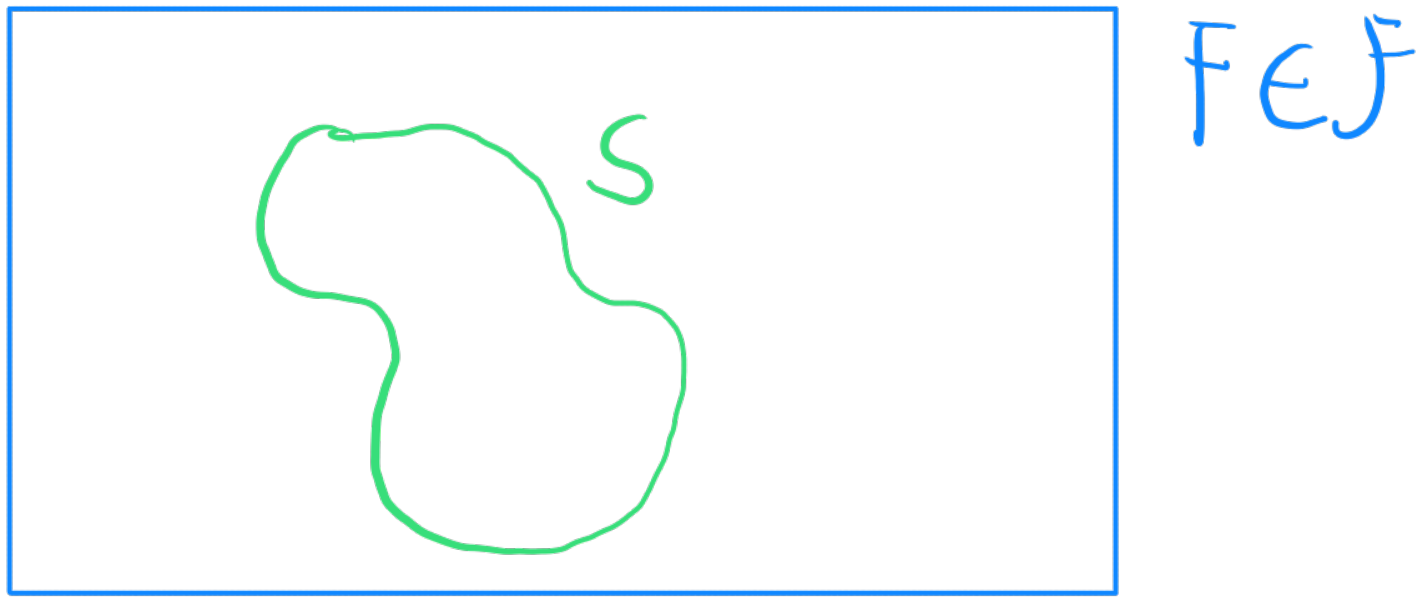
Construction schemes

We say $\mathcal{F}$ is a *construction scheme* if:

There is a partition $\mathcal{F} = \bigcup_{k\in\omega} \mathcal{F}_k$ such that:

1. $\mathcal{F}$ is cofinal.

Every $s \in [\omega_1]^{<\omega}$ is contained in an element of $\mathcal{F}$.



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3. All elements in each $\mathcal{F}_k$ have the same size.
4. The intersection of two elements of $\mathcal{F}_k$ is an initial segment of both.

5. For every $F \in \mathcal{F}_{k+1}$ there are $R(F)$ and $\{F_i \mid i < n\} \subseteq \mathcal{F}_k$ such that:

a. $\{F_i \mid i < n\}$ is a $\triangle$-system with root $R(F)$.

$(F_i \cap F_j = R(F) \text{ for } i \neq j)$

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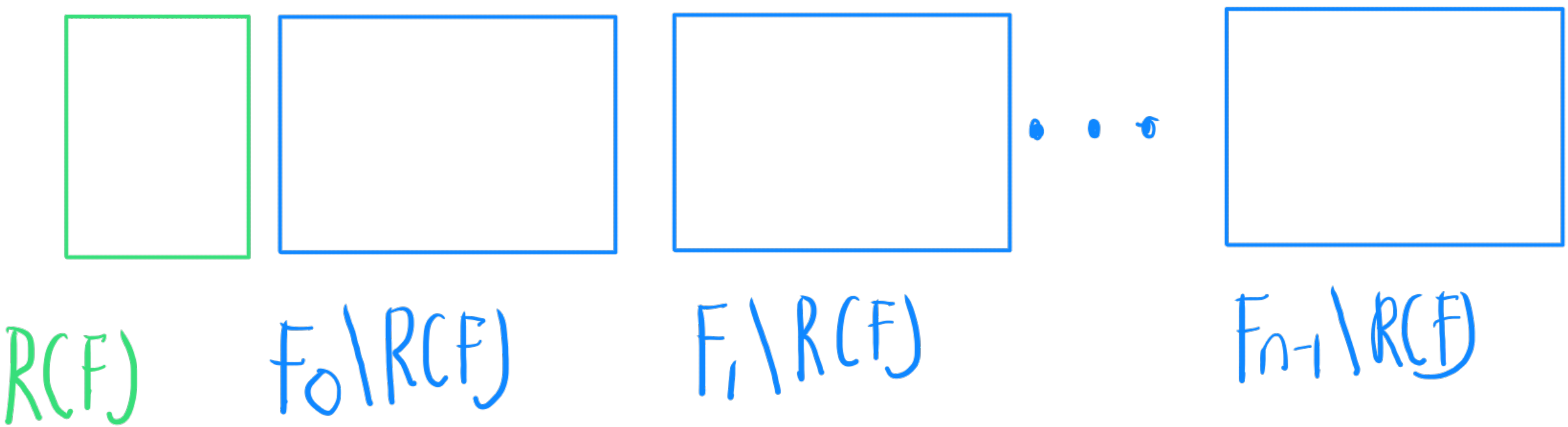
b. $R(F) < F_0 \setminus R(F) < \ldots < F_{n-1} \setminus R(F)$



a. $\{F_i \mid i < n\}$ is a Δ-system with root $R(F)$.

b. $R(F) < F_0 \setminus R(F) < \ldots < F_{n-1} \setminus R(F)$

c. $|R(F)|$ and n does not depend on F, only on k

$(F \in \mathcal{F}_{k+1})$

1) $\mathcal{F}$ is cofinal

2) $\mathcal{F}_1 = [\omega_1]^1$

3) If $E, F \in \mathcal{F}_k$, then $|E| = |F|$

4) If $E, F \in \mathcal{F}_k$, then $E \cap F \subsetneq E, F$

5) If $F \in \mathcal{F}_{k+1}$, then F is a Δ-system of elements of $\mathcal{F}_k$

Given $\mathcal{F}$ a construction scheme,
we define *its type* as $(m_k, n_k, r_k)_{k\in\omega}$:

1. $m_k = |E|$ for any $E \in \mathcal{F}_k$.
2. n_{k+1} is the number of pieces we glue at stage k.
3. r_{k+1} is the size of the root at stage k.

$$(m_k, n_k, r_k)$$

Size

Number of glued pieces

size of root

This numbers have the following properties:

1. $m_0 = 1.$
2. $n_k \geq 2.$
3. $m_k > r_{k+1}.$
4. $m_{k+1} = r_{k+1} + n_{k+1}(m_k - r_{k+1})$

We consider an additional property:

1. $m_0 = 1$.
2. $n_k \geq 2$.
3. $m_k > r_{k+1}$.
4. $m_{k+1} = r_{k+1} + n_{k+1}(m_k - r_{k+1})$.
5. For every $l \in \omega$, there are infinitely many k such that $r_l = k$.

Theorem (Todorcevic)

For every (m_k, n_k, r_k) as above, there is a Construction Scheme of that type

Capturing schemes

Let $\mathcal{F}$ be a construction scheme, $F \in \mathcal{F}_k$ and $a = \{\alpha_i \mid i < l\}$.

We say F *captures* a if:

1. $|a| = l \leq n_k$.

This means, F consists of at least l pieces.

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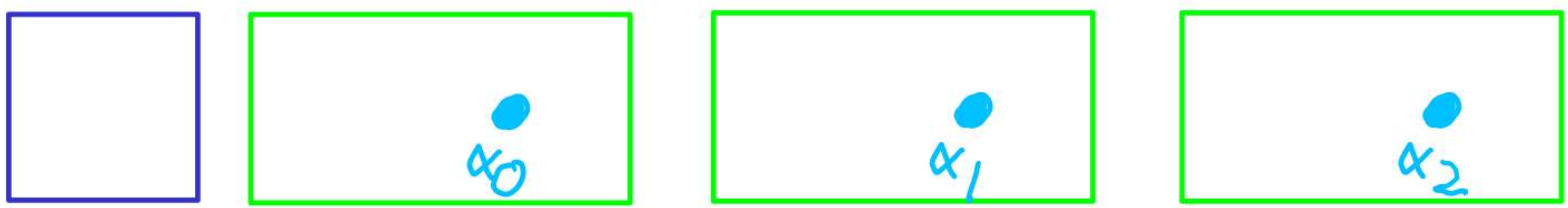
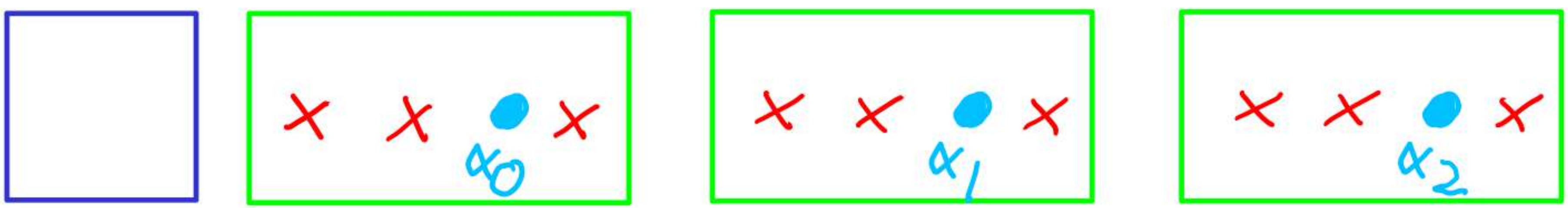
2. $\alpha_i \in F_i \setminus R(F).$

The i element of a is in the i piece of F.



1. $|a| = l \leq n_k.$

2. $\alpha_i \in F_i \setminus R(F).$ Moreover, they occupy the same position in each $F_i.$



In other words:

if α_0 is the j element of $F_0,$ then
α_1 is the j element of F_1 and
α_2 is the j element of F_2 and...

Definition

$\mathcal{F}$ is *n-capturing* if for every $S \in [\omega_1]^{\omega_1}$, there are $F \in \mathcal{F}$ and $a \in [S]^n$ such that F captures a.

Theorem (Chapital, G., Todorcevic)

$\diamondsuit$ implies that for every type and $n \in \omega$, there is an *n-capturing scheme.*

An application

CONTEMPORARY MATHEMATICS

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Partition Problems in Topology

Stevo Todorcevic

American Mathematical Society

In his book, Todorcevic develops an oscillation theory for an unbounded family. A similar theory can be developed using capture schemes.

In the book there are several constructions assuming $\mathfrak{b} = \omega_1$. Similar constructions can be done with capturing schemes.

This is relevant because the existence of capturing schemes is consistent with $\mathfrak{b}$ being arbitrary large.

Def

Let $f, g \in \omega^{\omega}$.

1) $f \leq g$ if $f(n) \leq g(n)$ for all $n \in \omega$

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Let $f, g \in \omega^\omega$.

1) $f \leq g$ if $f(n) \leq g(n)$ for all $n \in \omega$

2) $f \leq^* g$ if $f(n) \leq g(n)$ for all $n \in \omega$ except finitely many

Definition

We say X is an S-*space* if:

1. X is regular.
2. X is hereditary separable.
3. X is not Lindelöf.

Fix $\mathcal{F}$ a *2-capturing scheme*.

For every $\alpha \in \omega_1$, define $f_\alpha : \omega \to \omega$ as follows:

$$f_\alpha : \omega \longrightarrow \omega$$

Let $k \in \omega$, pick $F \in \mathcal{F}_k$ such that $\alpha \in F$.

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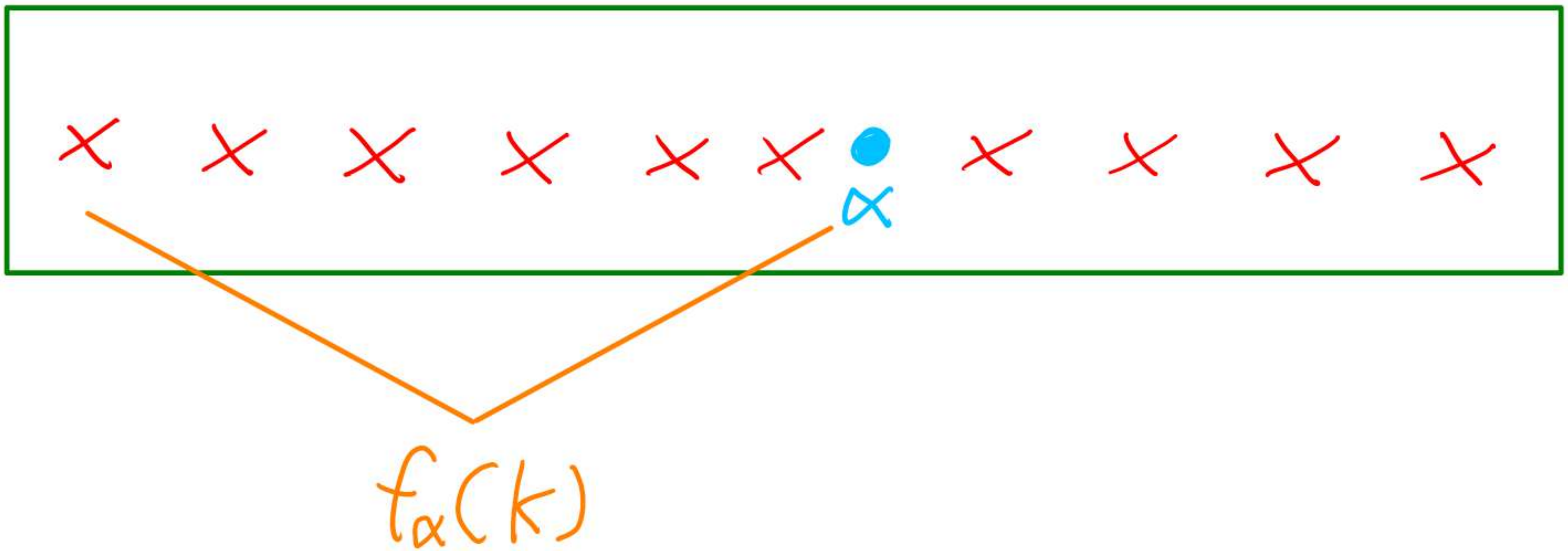
Let $k \in \omega$, pick $F \in \mathcal{F}_k$ such that $\alpha \in F$.



If α is the j-element of F, then $f_\alpha(k) = j$

This does not depend on F!

Define $\mathcal{B} = \{f_\alpha \mid \alpha \in \omega_1\}$.

Note that $\mathcal{B}$ is bounded ($f_\alpha(k) \leq m_k$ for every $k \in \omega$).

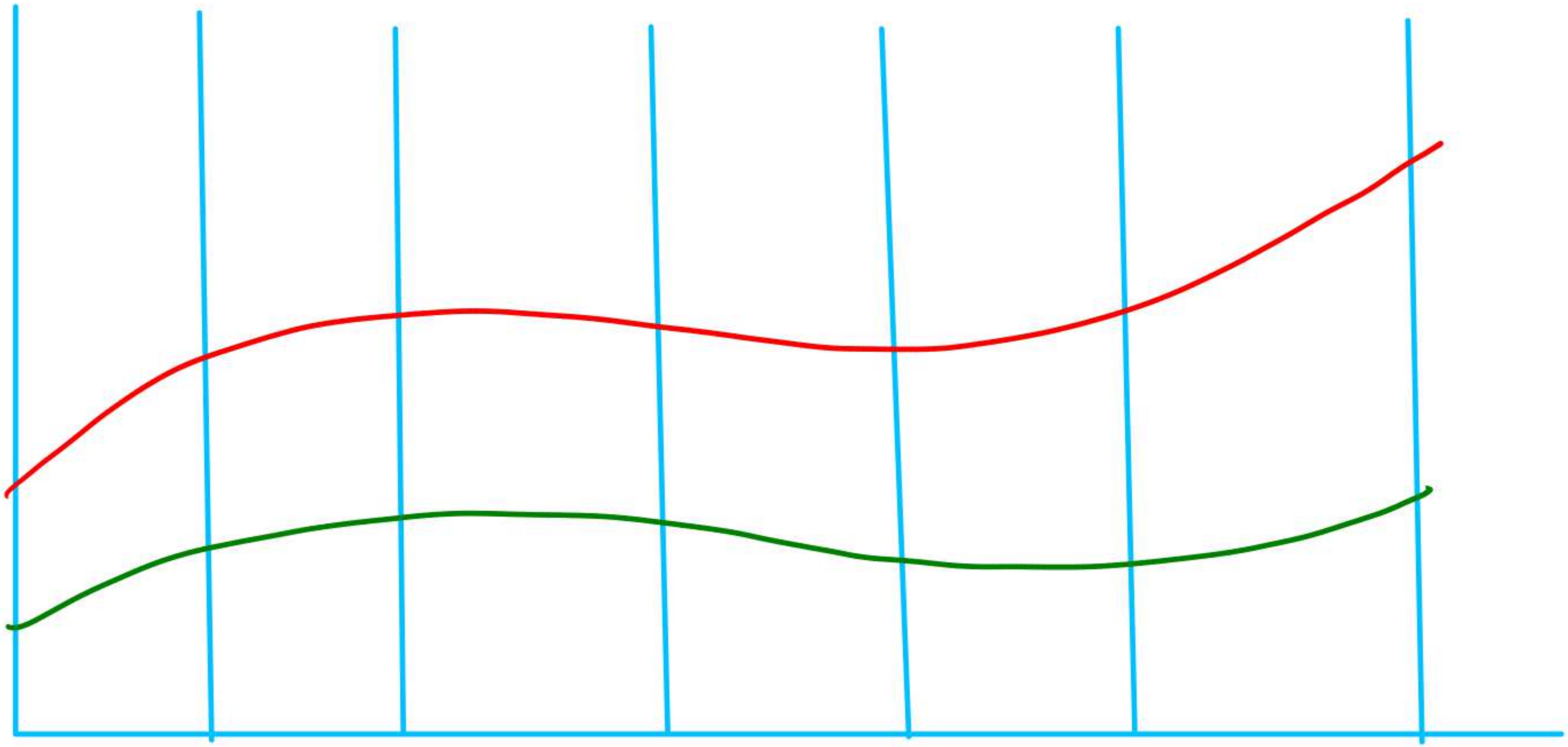
Note that if $\alpha < \beta$, then $f_\alpha \leq^* f_\beta$.

The 2-capturing property implies the following:

Proposition

For every $S \in [\omega_1]^{\omega_1}$, there are $\alpha < \beta \in S$ such that $f_\alpha \leq f_\beta$.

For $\beta \in \omega_1$, define $C(\beta) = \{f_\alpha \mid f_\alpha < f_\beta\}$.



Let τ be the topology on $\mathcal{B} \subseteq \omega^\omega$ refining the usual topology and declaring each $C(\alpha)$ clopen.

$$C(\alpha) = \{ f_\xi \mid f_\xi \leq f_\alpha \}$$

is closed in the metric topology

Theorem

$(\mathcal{B}, \tau)$ is an S-space.

It has a base of clopen sets and is right separated, so it remains to prove that it has no uncountable discrete set. Assume there is $D \in [\omega_1]^{\omega_1}$ such that $\{f_\alpha \mid \alpha \in D\}$ is discrete.

We can assume that there is $s \in \omega^{<\omega}$ such that f_α is the only element of $X \cap \langle s \rangle \cap C(\alpha)$ (for $\alpha \in D$).

By the previous proposition there are $\alpha < \beta \in D$ such that $f_\alpha < f_\beta$, so $f_\alpha \in X \cap \langle s \rangle \cap C(\beta)$, which is a contradiction.

Using construction/capturing schemes it is posible to build the following objects:

- Hausdorff gaps
- Luzin Jones AD families
- Aronszajn trees
- Suslin trees
- S-spaces
- Entangled sets
- Failures of Baumgartner axiom
- Suslin lattices
- Destructible gaps
- A sixth Tukey type
- Suslin towers
- and much more!

Thank you!