Characterized subgroup and its statistical variant related to some non-arithmetic sequences

Presentation by

Pratulananda Das

Department of Mathematics,
Jadavpur University,
West Bengal, India.

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Outline

Background

- An element x of an abelian group G is torsion if there exists $k \in \mathbb{N}$ such that $kx = 0$.
- Similarly, an element x of an abelian topological group G is [Braconnier, 1944]:
  - (i) *topologically torsion* if $n!x \to 0$;
  - (ii) *topologically p-torsion*, for a prime p, if $p^n x \to 0$.
- Consider the subgroups [Armacost, 1981]

$$\mathbb{T}_p = \{x \in \mathbb{T} : p^n x \to 0\} \quad \text{and} \quad \mathbb{T}! = \{x \in \mathbb{T} : n!x \to 0\}.$$

$\mathbb{T}_p$ is actually the Prufer group [Armacost] while the characterization of $\mathbb{T}!$ was obtained by [Borel, 1991]

Characterized subgroup

Definition

[for example see Dikranjan, Impieri, 2014] Let (a_n) be a sequence of integers, the subgroup

$$t_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ in } \mathbb{T}\}$$

of $\mathbb{T}$ is called a characterized (by (a_n)) subgroup of $\mathbb{T}$.

Arithmetic Sequence

A sequence of positive integers (a_n) is called an arithmetic sequence if

$$1 < a_1 < a_2 < a_3 < \ldots < a_n < \ldots \qquad \text{and } a_n | a_{n+1}$$

for every $n \in \mathbb{N}$

Statistical convergence

For $A \subseteq \mathbb{N}$, the upper and lower natural density of A is defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}. \tag{1}$$

We say that $d(A)$ exists if $\underline{d}(A) = \overline{d}(A)$.

Definition [Fast, Steinhaus, 1951]

A sequence (x_n) in $\mathbb{T}$ is said to converge to an $x_0 \in \mathbb{T}$ statistically if for any $\varepsilon > 0$, $d(\{n \in \mathbb{N} : \|x_n - x_0\| \geq \varepsilon\}) = 0$.

Statistically characterized subgroups

s-torsion element

[Dikranjan, Das, Bose, 2020] Let (a_n) be a sequence of integers. An element x in $\mathbb{T}$ is called topologically s-torsion element if $a_n x \to 0$ s-statistically in $\mathbb{T}$.

s-characterized Subgroup [Dikranjan, Das, Bose, 2020]

Let (a_n) be a sequence of integers, the subgroup

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ statistically in } \mathbb{T}\}$$

of $\mathbb{T}$ is called an s-characterized (by (a_n)) subgroup of $\mathbb{T}$.

Fact

[Dikranjan, Impieri, 2014] For any arithmetic sequence (a_n) and $x \in \mathbb{T}$, we can build a unique sequence of integers (c_n), where $0 \leq c_n < q_n$, such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \tag{2}$$

and $c_n < q_n - 1$ for infinitely many n.

Support

[Dikranjan, Impieri, 2014] For $x \in \mathbb{T}$ with canonical representation (2), we define

- $supp(x) = \{n \in \mathbb{N} \ : \ c_n \neq 0\}$;
- $supp_q(x) = \{n \in \mathbb{N} \ : \ c_n = q_n - 1\}$.

Clearly $supp_q(x) \subseteq supp(x)$.

For two subsets A, B of $\mathbb{N}$, we will write $A \subseteq^s B$ if $d(A \setminus B) = 0$ and $A =^s B$ if $d(A \triangle B) = 0$.

Outline

Some important observations regarding characterized subgroups [Dikranjan, Impieri, 2014]

Borel Complexity

For any sequence of integers (a_n), the subgroup $t_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$.

Cardinality

For an arithmetic sequence (a_n), the subgroup $t_{(a_n)}(\mathbb{T})$ is countable if and only if the sequence of ratios $(\frac{a_{n+1}}{a_n})$ is bounded.

Characterization for being a topologically torsion element

Let, (a_n) be an arithmetic sequence and $x \in \mathbb{T}$. Then x is a topologically torsion element (i.e., $x \in t_{(a_n)}(\mathbb{T})$) if and only if either $supp(x)$ is finite or if $supp(x)$ is infinite then for all infinite $A \subseteq \mathbb{N}$ the following holds:

(a) If A is q-bounded, then:

(a1) If $A \subseteq^* supp(x)$, then $A+1 \subseteq^* supp(x)$, $A \subseteq^* supp_q(x)$ and $\lim\limits_{n\in A} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$.
Moreover, if $A+1$ is q-bounded, then $A+1 \subseteq^* supp_q(x)$.

(a2) If $A \cap supp(x)$ is finite then $\lim\limits_{n\in A} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$.
Moreover, if $A+1$ is q-bounded, then $(A+1) \cap supp(x)$ is finite as well.

(b) If A is q-divergent then $\lim\limits_{n\in A} \frac{c_n}{q_n} = 0$ in $\mathbb{T}$.

Some important observations regarding s-characterized subgroups

Borel Complexity and Cardinality [Dikranjan, Das, Bose, 2020]

- For any sequence of integers (a_n), the subgroup $t^s_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.
- For any arithmetic sequence (a_n), we have $|t^s_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.
- For any arithmetic sequence (a_n), $t^s_{(a_n)}(\mathbb{T}) \supsetneq t_{(a_n)}(\mathbb{T})$.

Characterization for being an s-torsion element [Das, Ghosh, 2021]

Let, (a_n) be an arithmetic sequence and $x \in \mathbb{T}$. Then x is a topologically s-torsion element (i.e., $x \in t^s_{(a_n)}(\mathbb{T})$) if and only if either $d(supp(x)) = 0$ or if $\overline{d}(supp(x)) > 0$, then for all $A \subseteq \mathbb{N}$ with $\overline{d}(A) > 0$ the following holds:

(a) If A is q-bounded, then:

(a1) If $A \subseteq^s supp(x)$, then $A+1 \subseteq^s supp(x)$, $A \subseteq^s supp_q(x)$ and there exists $A' \subseteq A$ with $d(A \setminus A') = 0$ such that $\lim\limits_{n \in A'} \frac{c_{n+1}+1}{q_{n+1}} = 1$ in $\mathbb{R}$.
Moreover, if $A+1$ is q-bounded, then $A+1 \subseteq^s supp_q(x)$.

(a2) If $d(A \cap supp(x)) = 0$, then there exists $A' \subseteq A$ with $d(A \setminus A') = 0$ such that $\lim\limits_{n \in A'} \frac{c_{n+1}}{q_{n+1}} = 0$ in $\mathbb{R}$.
Moreover, if $A+1$ is q-bounded, then $d((A+1) \cap supp(x)) = 0$ as well.

(b) If A is q-divergent, then $\lim\limits_{n \in B} \frac{c_n}{q_n} = 0$ for some $B \subseteq A$ with $d(A \setminus B) = 0$.

Motivation I

A non-arithmetic sequence (ζ_n) was in fact considered in [Dikranjan, Kunen, 2007] for the sequence $(n!)$ as follows:

$$1, 2, 4, 6, 12, 18, 24, \ldots, n!, 2{\cdot}n!, 3{\cdot}n!, \ldots, n{\cdot}n!, (n+1)!, \ldots \quad (3)$$

Well known observation [Dikranjan, Kunen, 2007]

$t_{(\zeta_n)}(\mathbb{T}) = \mathbb{Q}/\mathbb{Z}$, i.e., $t_{(\zeta_n)}(\mathbb{T})$ coincides with the torsion subgroup of $\mathbb{T}$.

Open Problem 1 [Dikranjan, Das, Bose, 2020]

Compute $t^s_{(\zeta_n)}(\mathbb{T})$. Is it countable? Is it distinct from $\mathbb{Q}/\mathbb{Z}$?

Motivation II

Taking inspiration from the construction (3), we define the following general class of non-arithmetic sequences of integers for an arithmetic sequence (a_n).

Definition of (d_n)

Let (d_n) be an increasing sequence of integers formed by the elements of the set,

$$\{ra_k \ : \ 1 \leq r < b_{k+1}\}. \tag{4}$$

Note that for $a_n = n!$ corresponding non-arithmetic sequence (d_n) coincides with the sequence (ζ_n).

Motivation III

Problem formulation

Let (a_n) be an arithmetic sequence. Describe the subgroup $t_{(d_n)}(\mathbb{T})$ and $t^s_{(d_n)}(\mathbb{T})$.

Outline

Main Results related to $t_{(d_n)}(\mathbb{T})$ I

Theorem 1

For any arithmetic sequence (a_n), $x \in t_{(d_n)}(\mathbb{T})$ if and only if $supp_{(a_n)}(x)$ is finite.

Corollary 1A

For any arithmetic sequence (a_n), the following holds:

(i) $t_{(d_n)}(\mathbb{T}) = \bigcup\limits_{n=1}^{\infty} \left\langle \frac{1}{a_n} \right\rangle$.

(ii) $t_{(d_n)}(\mathbb{T})$ is countable. In particular, $t_{(d_n)}(\mathbb{T})$ is an F_σ subgroup of $\mathbb{T}$.

Main Results related to $t_{(d_n)}(\mathbb{T})$ II

Corollary 1B

For an arithmetic sequence of integers (a_n) the following conditions are equivalent:

(i) (a_n) is not b-bounded.

(ii) $|t_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.

(iii) $t_{(a_n)}(\mathbb{T}) \neq t_{(d_n)}(\mathbb{T})$.

(iv) $|t_{(a_n)}(\mathbb{T}) \setminus t_{(d_n)}(\mathbb{T})| = \mathfrak{c}$.

(v) $t_{(a_n)}(\mathbb{T})$ is not torsion.

Main Results related to $t_{(d_n)}(\mathbb{T})$ III

Theorem 2

For any arithmetic sequence (a_n),

$$t_{(d_n)}(\mathbb{T}) = t_{(a_n)}(\mathbb{T}) \cap \mathbb{Q}/\mathbb{Z} = t\big(t_{(a_n)}(\mathbb{T})\big).$$

For an arithmetic sequence of integers (a_n), we write

- $n_p(\underline{a}) = \infty$ precisely when p appears as divisor of b_n for infinitely many n
- $n_p(\underline{a}) = 0$ precisely when p does not divide b_n for any n.

Corollary 2A

For any arithmetic sequence (a_n), the subgroup $t_{(d_n)}(\mathbb{T})$ is divisible if and only if $n_p(\underline{a}) = 0$ or ∞ for each $p \in \mathbb{P}$.

Main Results related to $t^s_{(d_n)}(\mathbb{T})$ I

Theorem 3

Let (a_n) be an arithmetic sequence such that for each $m \in \mathbb{N}$, $\lim\limits_{n\to\infty} \frac{\sum\limits_{i=0}^{m-1}(b_{n-i}-1)}{\sum\limits_{i=1}^{n}(b_i-1)} = 0$. Then $|t^s_{(d_n)}(\mathbb{T})| = \mathfrak{c}$.

Solution of Problem 1

Let (ζ_n) be the sequence defined in Eq (3). Then $|t^s_{(\zeta_n)}(\mathbb{T})| = \mathfrak{c}$.

Outline

Some Open Problems

Problem 1A

- For any arithmetic sequence of integers (a_n), compute cardinality of the subgroup $t^s_{(d_n)}(\mathbb{T})$.

Problem 1B

- Characterize the elements of the subgroup $t^s_{(d_n)}(\mathbb{T})$ solely based on their support.

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Thank you for your attention