The Shape of Compact Covers

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joint work with Ziqin Feng

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- Motivation - free topological groups
- Background - Tukey order
- Initial Structure
- Cofinal Structure & Counting
- Conclusion

Free Topological Groups

Set X. Free group on X

$F(X) =$ all reduced words $x_1^{\pm 1} x_2^{\pm 1} \dots x_n^{\pm 1}$,

with natural product and inverse.

Every function $f : X \to G$, where G a group,

has canonical extension to a homomorphism $Ff : F(X) \to G$,

$Ff(x_1^{\pm 1} x_2^{\pm 2} \dots x_n^{\pm 1}) = f(x_1)^{\pm 1} f(x_2)^{\pm 1} \dots f(x_n)^{\pm 1}$.

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Space X. Free topological group on X is $F(X)$

with coarsest topological group topology

making Ff continuous

for all continuous $f : X \to G$, where G is a topological group.

Topology of Free Topological Groups

Let $\mathbb{U} = (\mathcal{U}_n)_n$ be such that $\mathcal{U}_n$ is an open cover of X^n.

Let $U_n(\mathbb{U}) = \{x_n^{-\epsilon_n} \ldots x_1^{-\epsilon_1} y_1^{\epsilon_1} \ldots y_n^{\epsilon_n} :$
$(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in U \in \mathcal{U}_n, \epsilon_i = \pm 1\}$.

Let $U(\mathbb{U}) = \bigcup_n \bigcup_{\sigma \in S_n} U_{\sigma(1)}(\mathbb{U}) U_{\sigma(2)}(\mathbb{U}) \ldots U_{\sigma(n)}(\mathbb{U})$.

Theorem (Tkachenko)

The collection of all $U(\mathbb{U})$ as above is a neighborhood base for the identity **1** *in $F(X)$.*

Lemma

The natural maps $\pi_n : X^n \times \{\pm 1\}^n \to F(X)$ are continuous.

Their images cover $F(X)$.

X embeds as a closed set in $F(X)$.

Diversity of Free Topological Groups

Theorem (Ward)

There are $\mathfrak{c}$-many metric continua whose free topological groups are pairwise non-topologically isomorphic.

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Idea:

(1) Relate 'shape' of compact covers of X and $F(X)$.

(2) Show there are many shapes of
compact covers of separable metrizable spaces.

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Idea:

(1) Relate 'shape' of compact covers of X and $F(X)$.

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Theorem

There are $2^{\mathfrak{c}}$-many separable metrizable spaces whose free topological groups are pairwise non-homeomorphic.

P-Ordered Compact Covers

Directed set P. Space X.

A compact cover $\mathcal{K}_P = \{K_p : p \in P\}$ of X is P-ordered if $p \leq p'$ implies $K_p \subseteq K_{p'}$.

Observe: ω-ordered compact cover iff σ-compact.

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Three drawbacks:

- Difficult to make comparisons
- External not internal definition
- Miss important features

Directed Set Pairs

P - directed set, and P' any subset. Consider the pair (P', P).

Subset C of P is *cofinal* for P' (in P) if
for every $p' \in P'$ there is c from C such that $c \geq p'$.

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Space X - natural pairs $(X, \mathcal{K}(X))$ and $(\mathcal{F}(X), \mathcal{K}(X))$,

where

$\mathcal{K}(X) =$ all compact subsets of X ordered by $\subseteq$,

$\mathcal{F}(X) =$ all finite subsets of X, and

the 'X' in $(X, \mathcal{K}(X))$ means the singletons of X.

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Observe -

the cofinal sets for $(X, \mathcal{K}(X))$
are precisely the compact covers of X.

(Relative) Tukey Order

To compare two pairs, say (P', P) and (Q', Q)...

write $(P', P) \geq_T (Q', Q)$,

and say '(P', P) Tukey quotients to (Q', Q)'

if and only if there is a map $\phi : P \to Q$ which takes

subsets of P cofinal for P' to subsets of Q cofinal for Q'.

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If $(P', P) \geq_T (Q', Q)$ and $(Q', Q) \geq_T (P', P)$ then the pairs are said to be *Tukey equivalent*, denoted $(P', P) =_T (Q', Q)$.

Abbreviate (P, P) by P.

When $(X, \mathcal{K}(X)) \geq_T (Y, \mathcal{K}(Y))$

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$\iff \exists \, \phi : \mathcal{K}(X) \to K(Y)$

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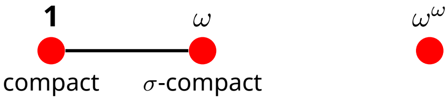
When $Q = \mathcal{K}(Y)$ then $(P', P) \geq_T (Q', Q)$
iff $\exists\, \phi : P \to \mathcal{K}(Y)$ which is
order-preserving and $\phi(P')$ cofinal for Q' in $\mathcal{K}(Y)$.

X has a P-ordered compact cover if and only if $P \geq_T (X, \mathcal{K}(X))$.

Initial Structure of $(M, \mathcal{K}(M))$ M separable metrizable



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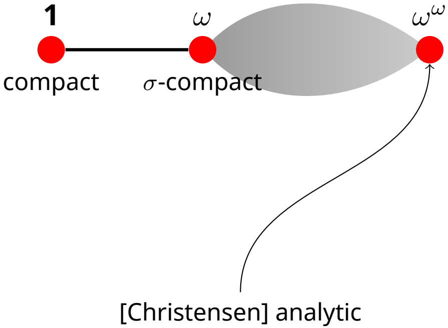

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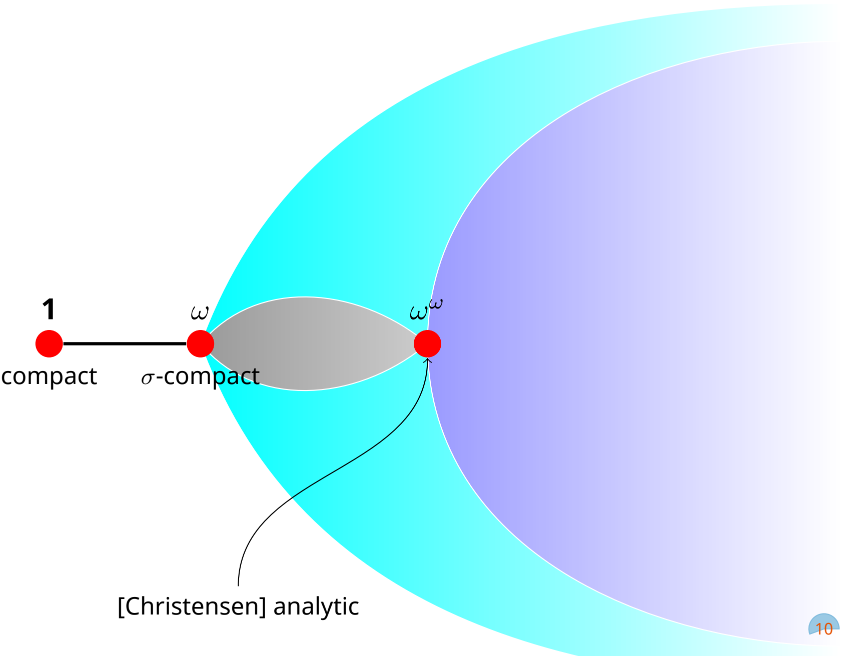

Menger Sets

A space is *Menger* if

for every sequence of open covers, $(\mathcal{U}_n)_n$,

one can select finite $\mathcal{V}_n \subseteq \mathcal{U}_n$

so that their union, $\bigcup_n \mathcal{V}_n$, cover.

A space is *strong Menger* if every finite power is Menger.

- σ-compact spaces are strong Menger,
- (ZFC) $\exists$ non-σ-compact strong Menger subsets of $\mathbb{R}$.

If $(M, \mathcal{K}(M)) \not\geq_T \omega^\omega$ then M Menger

Take open covers $(\mathcal{U}_n)_{n\in\omega}$. M is Lindelöf - $\mathcal{U}_n = \{U^n_m : m \in \omega\}$.

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We complete the proof by showing $\bigcup_n \bigcup \mathcal{V}_n$ covers.

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Well take any x in M, then $f_x(n_x) < f(n_x)$, so $x \in U^n_{f_x(n_x)} \in \mathcal{V}_{n_x}$.

If $(M, \mathcal{K}(M)) \geq_T \omega^\omega$ then M not Menger

Theorem

Let M be separable metrizable. Then:

(1) $(M, \mathcal{K}(M)) \not\geq_T \omega^\omega \Rightarrow$ *M is Menger, and*

(2) $(\mathcal{F}(M), \mathcal{K}(M)) \not\geq_T \omega^\omega \Rightarrow$ *M is strong Menger.*

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Question Does theorem hold for all Lindelöf spaces?

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★ $\omega^\omega =_T (\omega^\omega, \mathcal{K}(\omega^\omega))$ ★ ω^ω is not Menger

Lemma

Let M and N be separable metrizable spaces.

(1) $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$, *$M$ Menger $\Rightarrow$ N is also Menger.*

(2) $(\mathcal{F}(M), \mathcal{K}(M)) \geq_T (\mathcal{F}(N), \mathcal{K}(N))$ *and M is strong Menger*

$\Rightarrow$ *N is strong Menger.*

★ If $(M, \mathcal{K}(M)) \geq_T (N, \mathcal{K}(N))$ then can make N from M 'nicely'

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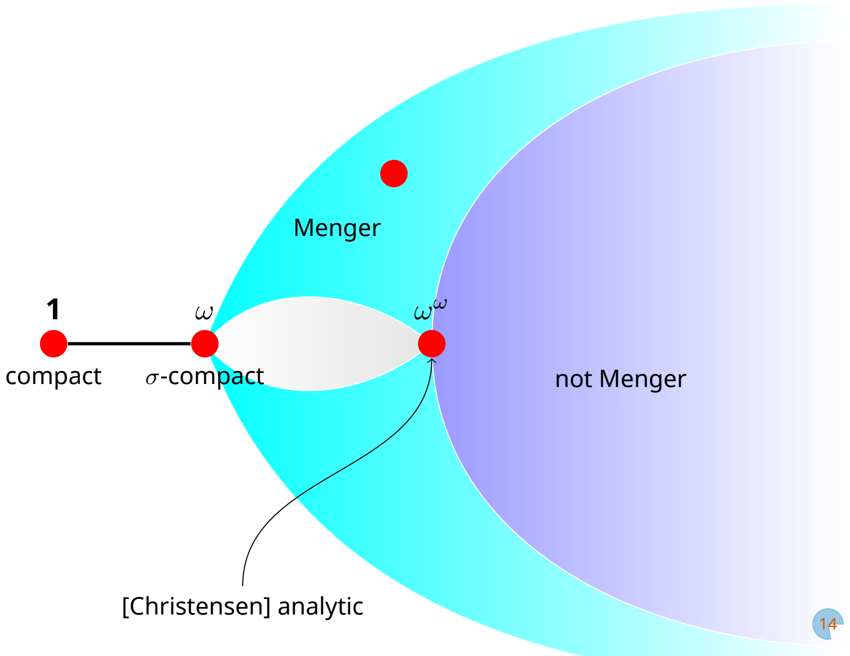
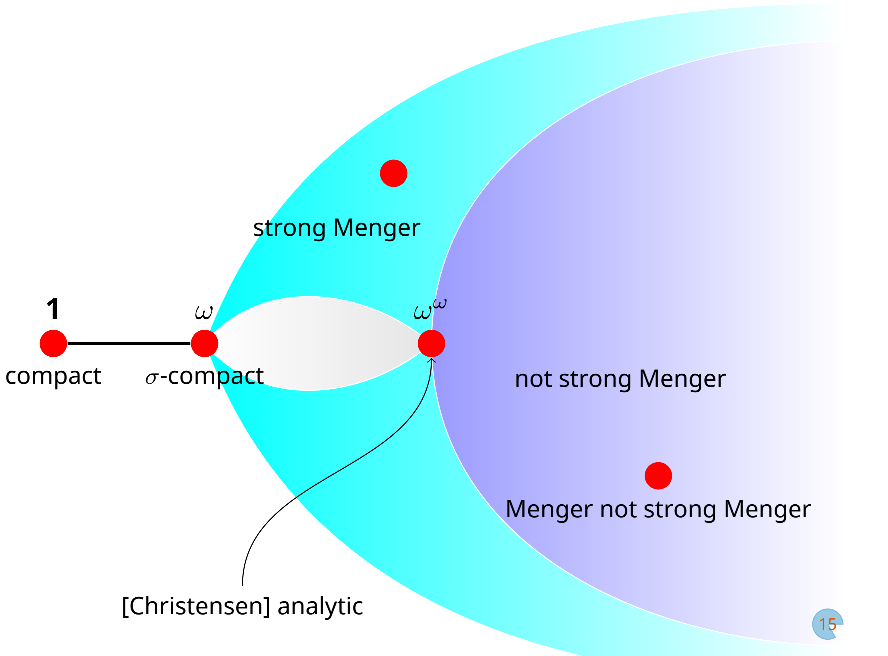
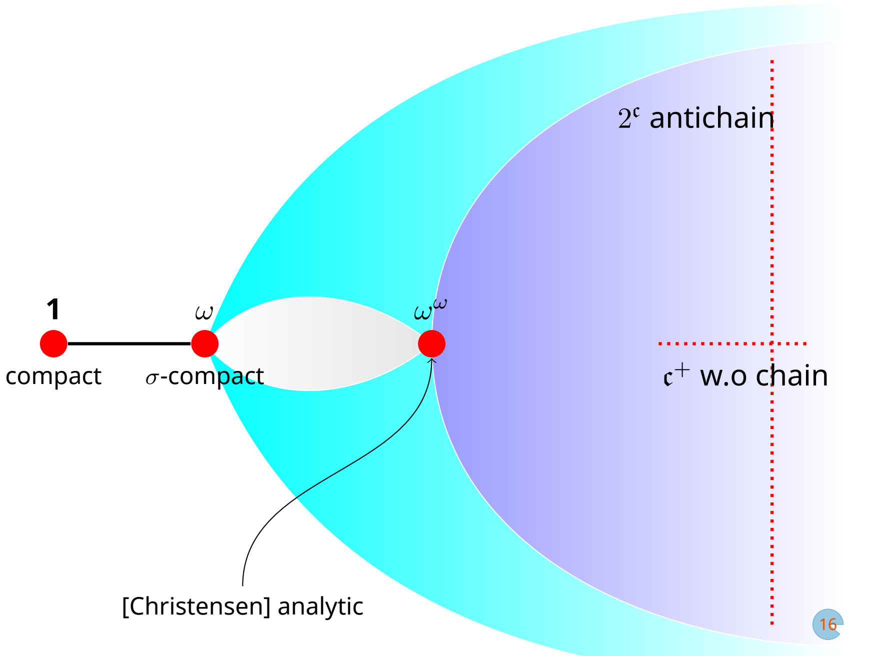


Initial Structure of $(\mathcal{F}(M), \mathcal{K}(M))$



Cofinal Structure of $(M, \mathcal{K}(M))$ and $(\mathcal{F}(M), \mathcal{K}(M))$



Counting Tukey Types

Theorem

There is a $2^{\mathfrak{c}}$-sized family, $\mathcal{M}$, of separable metrizable spaces such that

if M, N are distinct elements of $\mathcal{M}$ then

$(M, \mathcal{K}(M)) \not\geq_T (N, \mathcal{K}(N))$ *and* $(\mathcal{F}(M), \mathcal{K}(M)) \not\geq_T (\mathcal{F}(N), \mathcal{K}(N))$.

Theorem

If $2^{\mathfrak{b}} > \mathfrak{c}$ then there is a family $\mathcal{S}$ of $2^{\mathfrak{b}}$-many strong Menger sets such that

$(\mathcal{F}(M), \mathcal{K}(M)) \neq_T (\mathcal{F}(N), \mathcal{K}(N))$ *for distinct M and N from $\mathcal{S}$.*

Application to Free Topological Groups

Theorem

If X is not countably compact then

$$(\mathcal{F}(X), \mathcal{K}(X)) =_T (\mathcal{F}(F(X)), \mathcal{K}(F(X))).$$

Example

There is a $2^{\mathfrak{c}}$-sized family, $\mathcal{M}$ of separable metrizable spaces such that

if M, N are distinct elements of $\mathcal{M}$ then:

(1) $F(M)$ does not embed as a closed set in $F(N)$ and

(2) $F(M)$ is not the continuous image of $F(N)$.

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If $2^{\mathfrak{b}} > \mathfrak{c}$ there is a $2^{\mathfrak{b}}$-sized family, $\mathcal{S}$, of strong Menger sets such that

if M and N are distinct elements of $\mathcal{S}$

then $F(M)$ and $F(N)$ are not homeomorphic.

Thank you!

And please answer my questions :-)

- If X Lindelöf and $(X, \mathcal{K}(X)) \geq_T \omega^\omega$ then is X not Menger?
- (ZFC) Are there at least $2^{\mathfrak{b}}$-many Tukey inequivalent $(\mathcal{F}(M), \mathcal{K}(M))$ pairs where M is strong Menger?
  Are there at least $2^{\mathfrak{d}}$-many Tukey inequivalent $(M, \mathcal{K}(M))$ pairs where M is Menger?
- (ZFC) Is $2^{\mathfrak{d}}$ an upper bound on the number of Tukey inequivalent $(M, \mathcal{K}(M))$ pairs where M is Menger?
- Is it consistent that there are strictly fewer, up to Tukey equivalence, pairs $(\mathcal{F}(M), \mathcal{K}(M))$ where M is strong Menger than $(M, \mathcal{K}(M))$ pairs where M is Menger?

The Shape of Compact Covers by Feng & Gartside (arxiv:2401.00817) JSL (to appear)

Also see: *Menger and consonant sets in the Sacks model*

by Haberl, Szewczak & Zdomskyy (arxiv:2406.05457)