More on hyperspaces of knots

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Nadler's problem (1978):

Characterize the Vietoris hyperspace $\mathcal{S}(\mathbb{R}^n)$, $n > 1$, of simple closed curves.

Well-known properties of $\mathcal{S}(\mathbb{R}^n)$:

1. The exact absolute Borel class: $F_{\sigma\delta}$,
2. contains a copy of $c_0 = \{(x_k) \in \mathbb{R}^\infty : \lim x_k = 0\}$ as a closed subset,
3. is of the first category in itself,
4. $\mathcal{S}(\mathbb{R}^2)$ is arcwise connected and homogeneous (*any two simple closed curves in $\mathbb{R}^2$ are isotopic by an ambient isotopy of $\mathbb{R}^2$.*)

Properties (3) and (4) hold in the PL-category for the hyperspace $\mathcal{S}_P(\mathbb{R}^2)$ of polygonal planar simple closed curves.

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Properties (3) and (4) hold in the PL-category for the hyperspace $\mathcal{S}_P(\mathbb{R}^2)$ of polygonal planar simple closed curves.

Denote:
$\mathcal{K} := \mathcal{S}(\mathbb{R}^3)$,
$\mathcal{K}_P := \mathcal{S}_P(\mathbb{R}^3)$ = the hyperspace of polygonal knots,
$\mathcal{K}_T :=$ the hyperspace of tame knots.

Clearly, $\mathcal{K}_P \subset \mathcal{K}_T \subset \mathcal{K}$.

Proposition (2023)

1. *$\mathcal{S}_P(\mathbb{R}^2)$ and $\mathcal{K}_P$ are σ − compact*
2. *$\mathcal{S}_P(\mathbb{R}^2)$ and $\mathcal{K}_P$ contain a closed copy of*

$$\sigma = \{(x_j) \in [0,1]^\infty : x_j = 0 \textit{ for almost all } j\}$$

(an f.d.capset)

Proof. There is an embedding $f : [0,1]^\infty \to C(\mathbb{R}^2)$ such that

$$f((x_j)_{j=1}^\infty) \in \mathcal{S}_P(\mathbb{R}^2) \quad \text{if and only if} \quad (x_m)_{m=1}^\infty \in \sigma.$$

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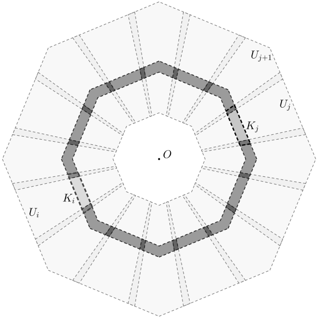
For the simple closed polygon P with vertices $e^{i\frac{2\pi j}{m}}$, $j = 1, \dots, m$ consider the annulus neighborhood of P:
$U = \{tz : z \in P,\ 1 - \frac{6}{n} < t < 1 + \frac{6}{n}\}$,

and $\langle U_1, \dots, U_n \rangle$ a Vietoris basic neighborhood in $C(\mathbb{R}^2)$ of P, with polyhedral open sectors U_k covering U.



Theorem

If $\mathcal{C}$ is any PL-topological family of continua in $\mathbb{R}^2$ containing P, then the neighborhood $\langle U_1, \ldots, U_n \rangle \cap \mathcal{C}$ of P in $\mathcal{C}$ is contractible in itself to P.

Since $\mathcal{S}_P(\mathbb{R}^2)$ is homogeneous, we get

Corollary

$\mathcal{S}_P(\mathbb{R}^2)$ has a basis of open AR-sets (i.e., it is strongly locally contractible).

Similarly, we get the strong local contractibility of $\mathcal{S}(\mathbb{R}^2)$.

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In $\mathbb{R}^3$, replace P, U, U_k by $P \times \{0\}$, $\widetilde{U} = U \times (-\frac{1}{n}, \frac{1}{n})$, $\widetilde{U_k} = U_k \times (-\frac{1}{n}, \frac{1}{n})$

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A space X is *locally homogeneous* if for each pair of points $x, y \in X$ there exist open neighborhoods U of x and V of y, and a homeomorphism $f : U \to V$ such that $f(x) = y$.

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The hyperspaces $\mathcal{K}_T$ and $\mathcal{K}_P$ are locally homogeneous.

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This is because

a knot C is tame if and only if there exist a neighborhood W of C and a homeomorphism $f : W \to \widetilde{U}$ such that $f(C) = P \times \{0\}$.

An analogue property holds in the PL-category:

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Theorem

1. *$\mathcal{S}_P(\mathbb{R}^2)$ and $\mathcal{K}_P$ are σ-compact, strongly countable-dimensional ANRs.*
2. *Each compact subset of $\mathcal{S}_P(\mathbb{R}^2)$ and of $\mathcal{K}_P$ is a Z-set.*

Proof. (1) Sakai proved that the Vietoris hyperspace $Pol(X)$ of all connected compact polyhedra in a compact convex subset X of $\mathbb{R}^n$, $n > 1$, is $\cong \sigma$, so it is σ-compact, strongly countable-dimensional.

Hence, the hyperspace $Pol(\mathbb{R}^n)$ of connected compact polyhedra in $\mathbb{R}^n$ also is σ-compact and strongly countable-dimensional.

Since $\mathcal{S}_P(\mathbb{R}^2)$ and $\mathcal{K}_P$ are σ-compact subsets of $Pol(\mathbb{R}^3)$, they are strongly countable-dimensional locally contractible spaces, so they are ANRs.

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Proposition

$\mathcal{S}_P(\mathbb{R}^2)$, $\mathcal{S}(\mathbb{R}^2)$, $\mathcal{K}$, $\mathcal{K}_P$, and $\mathcal{K}_T$ are arcwise connected.

Proof. Classical for $\mathcal{S}_P(\mathbb{R}^2)$, $\mathcal{S}(\mathbb{R}^2)$.
In the case of knots, it is enough to exhibit, for any knot C, a PL-isotopy on $\mathbb{R}^3$ transferring C into $\langle \widetilde{U_1}, \ldots, \widetilde{U_n} \rangle$.

The following facts are well known.

Fact

The action of the autohomeomorphism group H of $\mathbb{R}^3$ on $\mathcal{K}_T$ has countably many orbits which coincide with orbits of polygonal knots.

Fact

$\mathcal{K}_T$ is absolute Borel.

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An infinite-dimensional space X is called an *infinite-dimensional Cantor manifold* if no finite-dimensional closed subset separates X.

If no weakly infinite-dimensional closed subset separates X, then X is *strongly infinite-dimensional Cantor manifold*.

The Hilbert cube $[0,1]^\infty$ and $c_0 \cong \{(x_j) \in [0,1]^\infty : \lim_j x_j = 0\}$ are strongly infinite-dimensional Cantor manifolds.

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2. *The hyperspaces* $\mathcal{S}_P(\mathbb{R}^2)$ *and* $\mathcal{K}_P$ *are infinite-dimensional Cantor manifolds.*

In the proof, we use

- the existence of copies of c_0 in $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{K}_T$ and of copies of σ in $\mathcal{S}_P(\mathbb{R}^2)$ and $\mathcal{K}_P$,
- the local homogeneity of the hyperspaces,

and the following facts.

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Let H^+_{PL} denote the subgroup of H consisting of orientation preserving PL-autohomeomorphisms.

Fact

Knots C, D belong to the same orbit of the group H^+_{PL} if and only if C and D have the same PL-isotopy type.

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Each orbit of H^+_{PL} on $\mathcal{K}_T$ is dense in $\mathcal{K}_T$.

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Arcs and triods in $\mathbb{R}3$

Behave like knots:

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If $\mathcal{A} \subset C(\mathbb{R}3)$ contains an arc $A = [0,1] \times \{0\}$, then $\langle \widetilde{U_1}, \ldots, \widetilde{U_n} \rangle \cap a(\mathbb{R}^3)$ is contractible, where $\widetilde{U_1}, \ldots, \widetilde{U_n}$ is a chain of open polyhedral 3-cells

Theorem

$a_P(\mathbb{R}^3)$ and $t_P(\mathbb{R}^3)$ are σ-compact, strongly countable-dimensional ANRs which are infinite-dimensional Cantor manifolds. $a_T(\mathbb{R}^3)$,$t_T(\mathbb{R}^3)$ have open basis of AR-sets and are strongly infinite-dimensional Cantor manifolds.

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