Interval maps with dense periodicity

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(joint work with Jozef Bobok, Jernej Činč and Serge Troubetzkoy)

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Basic setting

1. $I := [0,1]$ denote the unit interval,
2. $C(I)$ continuous interval maps,
3. $\rho(f,g) := \sup_{x\in I} |f(x) - g(x)|$ the uniform metric on $C(I)$
4. λ – the Lebesgue measure on $I = [0,1]$.
5. $Per(f) := \{x \in I : \exists n \in \mathbb{N} \text{ s.t. } f^n(x) = x\}$ - periodic points of f,
6. $CP := \{f \in C(I) : \overline{Per(f)} = I\}$ - maps with dense periodicity.
7. $\overline{CP}$ the closure of CP with respect to ρ.
8. $C_\lambda(I) = \{f \in C(I);\ \forall\ A \subset [0,1], A \text{ Borel} :\ \lambda(A) = \lambda(f^{-1}(A))\}$..
   Note that $C_\lambda(I) = \overline{C_\lambda(I)}$.
9. **a property P is typical in $(C_\lambda(I), \rho) \equiv$ the set of all maps with the property P is residual, maps bearing a typical property are called generic.**

Main questions

We study the following questions for maps from CP and $\overline{CP}$:

1. Are characterizations of maps in CP and $\overline{CP}$ possible?
2. If not, what can be said about typical maps there?
3. How many distinct conjugacy classes

$$\{\psi^{-1} \circ f \circ \psi \in C(I) : \psi \text{ is a homeomorphism of } I\}$$

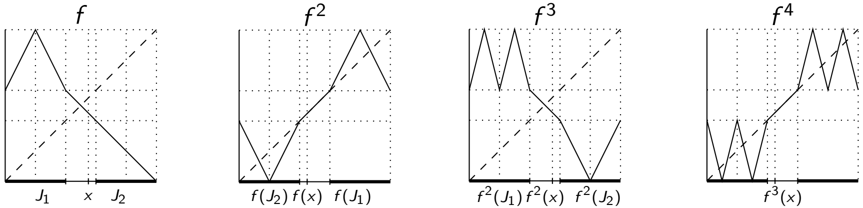
are there?

- Related question: recently we proved with Činč that the inverse limit of any generic map in $C_\lambda(I)$ is the pseudo-arc (using techniques from Minc-Transue (1991)) and constructed a parameterized family of planar homeomorphisms with pseudo-arc attractors. Are these maps all topologically conjugate?

Barge-Martin Theorem (1985) - decomposition in CP

Suppose $f \in CP$. The following holds:

1. There is a collection (perhaps finite or empty) $\mathcal{J} = \{J_1, J_2, \dots\}$ of closed subintervals of I with mutually disjoint interiors, such that for each $J \in \mathcal{J}$ we have $f^2(J) = J$.
2. If $x \in (0,1) \setminus \bigcup_{i \geqslant 1} \mathrm{int} J_i$, then $f^2(x) = x$.
3. For each $J \in \mathcal{J}$, the map $f^2|_J$ is topologically mixing.



Relation between $C_\lambda(I)$ and CP

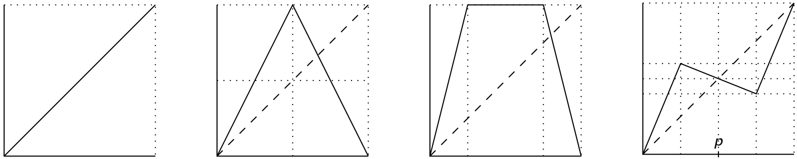
1. Let f be an interval map. The following conditions are equivalent.
    1. f has a dense set of periodic points, i.e., $\overline{\mathrm{Per}(f)} = I$.
    2. f preserves a nonatomic probability measure μ with $\mathrm{supp}\ \mu = I$.
    3. There exists a homeomorphism h of I such that $h \circ f \circ h^{-1} \in C_\lambda(I)$.
2. for $C_\mu(I)$, $\mathrm{supp}\ \mu = I$, nonatomic
    1. all topological *"typical properties"* can be translated to the complete metric space $(C_\mu(I), \rho)$

Some hints (works of J. Bobok, S. Troubetzkoy, 2020)

1. $C_\lambda(I)$ typical map f
   1. is weakly mixing with respect to λ,
   2. is topologically exact,
   3. satisfies the periodic specification property,
   4. has infinite topological entropy,
   5. has its graph of Hausdorff dimension = lower Box dimension = 1.
2. and its graph upper Box dimension = 2
   [J. Schmeling, R. Winkler, *Typical dimension of the graph of certain functions*, Monatsh. Math. **119** (1995), 303–320].
3. In joint works with Jernej (and Jozef and Serge) we extended this list even further:
   - inverse limit with f as bounding map is the pseudo-arc,
   - periodic points of any period form a Cantor set,
   - f has the shadowing property,
   - ...

Chain recurrence

1. any $x = x_0, \ldots, x_n = y$ where $n > 0$ and $|f(x_i) - x_{i+1}| < \varepsilon$ for $0 < i < n-1$ is ε-chain from x to y
2. x is *chain-recurrent* (for f) if for every $\varepsilon > 0$ there is an ε-chain from x to itself.
3. f is chain-recurrent is every x is chain-recurrent
4. f is chan-transitive if there is ε-chain between x, y for any $x, y \in X$ and any $\varepsilon > 0$.

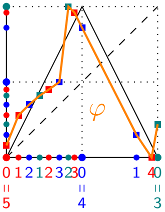


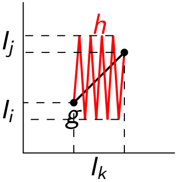
5. Remark: Every chain-recurrent map on connected space is chain-transitive.

Chain-recurrence

Theorem

A map $f \in C(I)$ is chain-recurrent if and only if $f \in \overline{CP}$.





Consequences of chain-recurrence

Theorem (BČOT, 2024)

A map $f \in C(I)$ is chain-recurrent if and only if $f \in \overline{CP}$.

Theorem (Block-Coven, 1986)

Chain recurrent f is either s.t. $f^2 = \mathrm{id}$ or f^2 is turbulent. Turbulent maps have h_{top} at least $\log 2$.

Corollary (BČOT, 2024)

The following hold:

1. *The maps id and up to conjugacy the map $1 - \mathrm{id}$ are the only maps with zero topological entropy in $\overline{CP}$.*
2. *If $f \in \overline{CP}$ such that $h_{\mathrm{top}}(f) > 0 \implies f^2$ turbulent. Therefore, either $h_{top}(f) = 0$ or $h_{top}(f) \geq \log 2/2$.*

Topological exactness (leo)

1. A map f is leo or topologically exact if for every non-empty open $U \subset I$ there exists $n \in \mathbb{N}$ such that $f^n(U) = I$.

$$\text{leo} \implies \text{topologically mixing} \implies \text{transitive}$$

Theorem (Bobok, Troubetzkoy, 2020)

Set of leo maps is residual in the set of continuous Lebesgue measure-preserving interval maps $C_\lambda(I)$.

Theorem (BČOT, 2024)

Set of leo maps is open and dense in the set CP.

Corollary (BČOT, 2024)

Set of leo maps is residual in the set $\overline{CP}$ but is not open.

Conjugacy classes

Let $\mathcal{H}(I)$ denote the set of all homeomorphism (increasing or decreasing) of I. For $f \in \overline{CP}$ put

$$G_f := \{\psi^{-1} \circ f \circ \psi : \psi \in \mathcal{H}(I)\}.$$

All maps in G_f are chain recurrent, thus $\overline{G_f}$ is a subset of $\overline{CP}$.

Theorem (BČOT, 2024)

For every $f \in \overline{CP}$, the set G_f is nowhere dense in $\overline{CP}$.

Thus each union of countably many conjugacy classes is a meager set, so we have

Corollary (BČOT, 2024)

Any residual set $G \subset \overline{CP}$ contains uncountably many conjugacy classes.

Remark

Analogous results hold for CP and $C_\lambda(I)$.