# Higher dimensional compactness properties

Paul J. Szeptycki

Department of Mathematics and Statistics
York University
Toronto Canada

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Joint work with César Corral, Pourya Memarpanahi and others...

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Recall:

- If $X$ is compact then $\forall f \forall p$ ($f$ has a $p$-limit point).
- $X$ is countably compact iff $\forall f \exists p$ ($f$ has a $p$-limit point).

# Higher dimensional convergence

## Definition

Given a function $f : [\omega]^n \to X$ and $M \subseteq \omega$ we say that $f \restriction [M]^n$ converges to $x \in X$ if for every nbhd $U$ of $x$

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## Nash-Williams's Theorem

If $k \in \omega$ and $\mathcal{B} \subseteq [\omega]^{<\omega}$ a barrier, then for every $f : \mathcal{B} \to k$ there is $M$ infinite such that $f$ is constant on $\mathcal{B} \cap [M]^{<\infty}$.

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$\Leftrightarrow$ Compact metrizable spaces are $\mathcal{B}$-sequentially compact.

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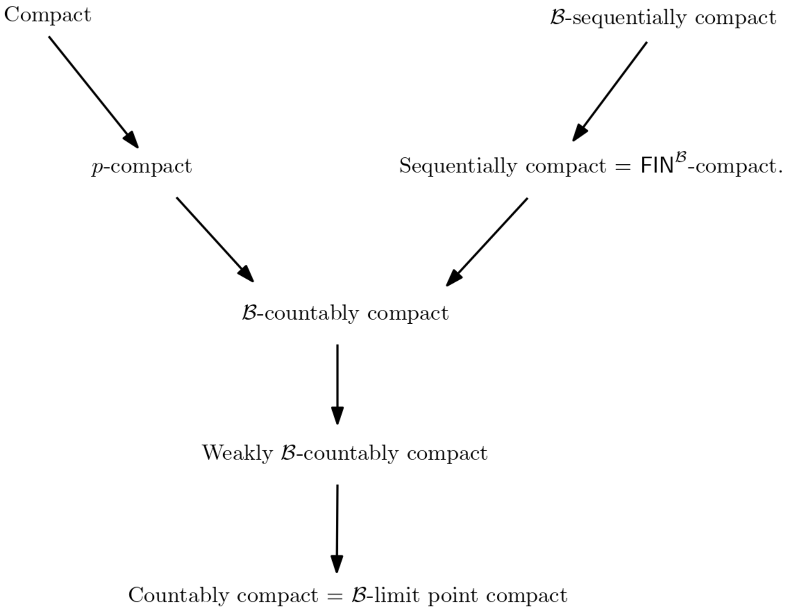
**Definition**

$X$ is $B$-countably compact if every $f : B \to X$ has a $p^{\mathcal{B}}$-limit point for some $p \in \omega^*$.

# Relationship between the different properties



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**Definition (Banakh-Dimitrova-Gutik)**

$X$ is doubly countably compact if for every $f : \omega^2 \to X$, there exists $(x_n : n \in \omega)$ and $x$ in $X$ such that $x_n = \lim_p(f(n, m) : m \in \omega)$ and $x = \lim_p(x_n)$.

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Doubly countably compact $\Rightarrow$ 2-countably compact.

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**Theorem (B-D-G)**

A topological semigroup $G$ contains an idempotent if and only if for some $x \in G$ the double sequence $(x^{m-n})$ has a double $p$-limit for some $p \in \omega^*$.

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**Question**

Are 2-countably compacts spaces doubly countably compact?

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2. A doubly countably compact subspace of $\beta\omega$ whose square is not countably compact

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- Conjecture: If $X \subseteq \beta\omega$ has size $< 2^{\mathfrak{c}}$ then $\beta\omega \setminus X$ is $\mathcal{B}$-countably compact for any barrier $\mathcal{B}$ (in ZFC).

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- Conjecture: If $X \subseteq \beta\omega$ has size $< 2^{\mathfrak{c}}$ then $\beta\omega \setminus X$ is $\mathcal{B}$-countably compact for any barrier $\mathcal{B}$ (in ZFC). (YES if there are $2^{\mathfrak{c}}$ many selective ultrafilters).

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THANK YOU

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