Hurewicz sets and products in the Laver model

Piotr Szewczak

Cardinal Stefan Wyszyński University in Warsaw, Poland

joint work with Lyubomyr Zdomskyy

The research was funded by the National Science Center, Poland
Weave-UNISONO call in the Weave programme
Project: *Set-theoretic aspects of topological selections* 2021/03/Y/ST1/00122

Covering properties

Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists\, U \in \mathcal{U})(F \subseteq U)$

Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists U \in \mathcal{U})(F \subseteq U)$

$\mathcal{O}$: all open covers $\quad$ Γ : all γ-covers $\quad$ Ω : all ω-covers

Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists\, U \in \mathcal{U})(F \subseteq U)$

$\mathcal{O}$: all open covers Γ : all γ-covers Ω : all ω-covers

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{A})$: $(\forall \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{A})$

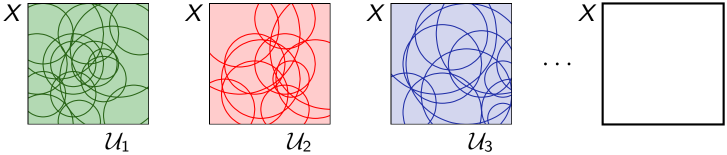
Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists U \in \mathcal{U})(F \subseteq U)$

$\mathcal{O}$: all open covers $\quad$ Γ : all γ-covers $\quad$ Ω : all ω-covers

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{A})$: $(\forall \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{A})$



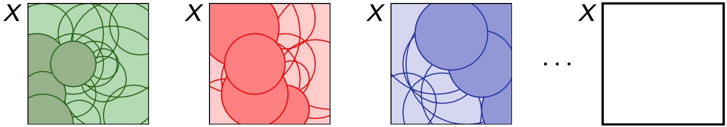
Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists\, U \in \mathcal{U})(F \subseteq U)$

$\mathcal{O}$: all open covers $\quad$ Γ : all γ-covers $\quad$ Ω : all ω-covers

$\mathrm{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{A})$: $(\forall \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{A})$



$\mathcal{F}_1 \subseteq \mathcal{U}_1$ $\quad$ $\mathcal{F}_2 \subseteq \mathcal{U}_2$ $\quad$ $\mathcal{F}_3 \subseteq \mathcal{U}_3$

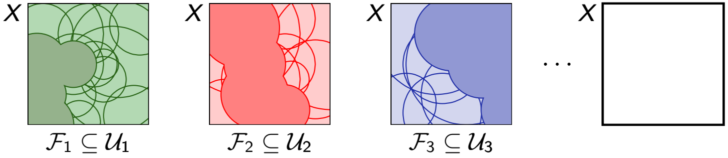
Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists\, U \in \mathcal{U})(F \subseteq U)$

$\mathcal{O}$: all open covers $\quad$ Γ : all γ-covers $\quad$ Ω : all ω-covers

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{A})$: $(\forall\, \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{A})$



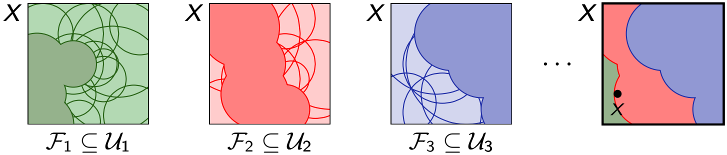
Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists\, U \in \mathcal{U})(F \subseteq U)$

$\mathcal{O}$: all open covers Γ : all γ-covers Ω : all ω-covers

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{A})$: $(\forall \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{A})$



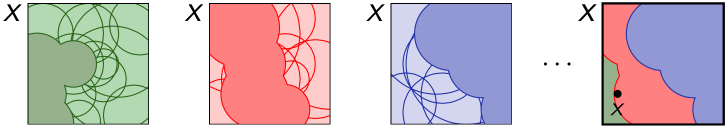
Covering properties

γ-cover: $\mathcal{U}$ is infinite and $(\forall x \in X)(x$ belongs to all but fin many $U \in \mathcal{U})$

ω-cover: $X \notin \mathcal{U}$ and $(\forall$ finite $F \subseteq X)(\exists\, U \in \mathcal{U})(F \subseteq U)$

$\mathcal{O}$: all open covers Γ : all γ-covers Ω : all ω-covers

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{A})$: $(\forall \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{A})$



$\mathcal{F}_1 \subseteq \mathcal{U}_1$ $\mathcal{F}_2 \subseteq \mathcal{U}_2$ $\mathcal{F}_3 \subseteq \mathcal{U}_3$

$$\begin{tikzcd}
 & \text{Hurewicz} & \text{Scheepers} & \text{Menger} & \\
\sigma\text{-compact} \arrow[r, shift right] & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) \arrow[l, shift right, bend right, "\times"'] \arrow[r, shift right] & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega) \arrow[l, shift right, bend right, "\times"'] \arrow[r] & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \arrow[r, shift right] & \text{Lindelöf} \arrow[l, shift right, bend right, "\times"'] \\
 & \mathfrak{b} & \mathfrak{d} & \mathfrak{d} &
\end{tikzcd}$$

Hurewicz

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $(\forall\,\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathcal{O})(\exists\,\text{fin }\mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots)(\{\bigcup\mathcal{F}_n : n\in\mathbb{N}\}\in\Gamma)$

Hurewicz

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $(\,\forall\,\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathcal{O}\,)(\,\exists\,\text{fin}\ \mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots)(\,\{\,\bigcup\mathcal{F}_n : n\in\mathbb{N}\,\}\in\Gamma\,)$

- Sierpiński set

Hurewicz

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $(\forall\,\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathcal{O})(\,\exists\,\text{fin}\ \mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots)(\,\{\,\bigcup\mathcal{F}_n : n\in\mathbb{N}\,\}\in\Gamma\,)$

- Sierpiński set
- $X\cup\mathrm{Fin}$, where $X=\{\,x_\alpha : \alpha<\mathfrak{b}\,\}\subseteq[\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, i.e., X is unbounded and $x_\alpha\leq^* x_\beta$ for $\alpha<\beta$

Hurewicz

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $(\forall\,\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathcal{O})(\exists\text{ fin }\mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots)(\{\bigcup\mathcal{F}_n : n\in\mathbb{N}\}\in\Gamma)$

- Sierpiński set
- $X\cup\mathrm{Fin}$, where $X=\{x_\alpha : \alpha<\mathfrak{b}\}\subseteq[\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, i.e., X is unbounded and $x_\alpha\leq^* x_\beta$ for $\alpha<\beta$
- when ground model sets became Hurewicz in the extension

Hurewicz

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $(\forall\,\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathcal{O})(\exists\text{ fin }\mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots)(\{\bigcup\mathcal{F}_n : n\in\mathbb{N}\}\in\Gamma)$

- Sierpiński set
- $X\cup\mathrm{Fin}$, where $X=\{x_\alpha : \alpha<\mathfrak{b}\}\subseteq[\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, i.e., X is unbounded and $x_\alpha\leq^* x_\beta$ for $\alpha<\beta$
- when ground model sets became Hurewicz in the extension

Theorem (Just–Miller-Scheepers–Szeptycki 1996)

If $X\subseteq\mathrm{P}(\mathbb{N})$ is Hurewicz and does not contain a copy of the Cantor set, then for each perfect set $P\subseteq\mathrm{P}(\mathbb{N})$, the set $X\cap P$ is meager in P.

Hurewicz

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $(\forall\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathcal{O})(\exists\,\text{fin}\ \mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots)(\{\bigcup\mathcal{F}_n : n\in\mathbb{N}\}\in\Gamma)$

- Sierpiński set
- $X\cup\mathrm{Fin}$, where $X=\{x_\alpha:\alpha<\mathfrak{b}\}\subseteq[\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, i.e., X is unbounded and $x_\alpha\leq^* x_\beta$ for $\alpha<\beta$
- when ground model sets became Hurewicz in the extension

Theorem (Just–Miller-Scheepers–Szeptycki 1996)

If $X\subseteq\mathrm{P}(\mathbb{N})$ is Hurewicz and does not contain a copy of the Cantor set, then for each perfect set $P\subseteq\mathrm{P}(\mathbb{N})$, the set $X\cap P$ is meager in P.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d}=\omega_1$, each productively Lindelöf space is productively Hurewicz.

Hurewicz

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $(\forall \mathcal{U}_1,\mathcal{U}_2,\ldots \in \mathcal{O})(\exists \text{fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \Gamma)$

- Sierpiński set
- $X \cup \mathrm{Fin}$, where $X = \{x_\alpha : \alpha < \mathfrak{b}\} \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, i.e., X is unbounded and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$
- when ground model sets became Hurewicz in the extension

Theorem (Just–Miller-Scheepers–Szeptycki 1996)

If $X \subseteq \mathrm{P}(\mathbb{N})$ is Hurewicz and does not contain a copy of the Cantor set, then for each perfect set $P \subseteq \mathrm{P}(\mathbb{N})$, the set $X \cap P$ is meager in P.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d} = \omega_1$, each productively Lindelöf space is productively Hurewicz.

Theorem (Chodounsky–Repovš–Zdomskyy 2015)

Let $F \subseteq [\mathbb{N}]^\infty$ be a filter. Then $\mathbb{M}(F)$ preserves unbounded sets from the ground model if and only if F is Hurewicz.

Products

γ-property: each open ω-cover contains a γ-cover

$$\begin{array}{ccccccccc} & & \text{Hurewicz} & & \text{Scheepers} & & \text{Menger} & & \\ \gamma\text{-property} & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) & \longrightarrow & \text{Lindelöf} \end{array}$$

Products

γ-property: each open ω-cover contains a γ-cover

$$\begin{array}{ccccccccc} & & \text{Hurewicz} & & \text{Scheepers} & & \text{Menger} & & \\ \gamma\text{-property} & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) & \longrightarrow & \text{Lindelöf} \end{array}$$

Theorem (Todorčević 1995)

For general topological spaces, there are γ-sets whose product is not Lindelöf.

Products

γ-property: each open ω-cover contains a γ-cover

$$\gamma\text{-property} \longrightarrow \underset{\text{Hurewicz}}{\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)} \longrightarrow \underset{\text{Scheepers}}{\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega)} \longrightarrow \underset{\text{Menger}}{\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})} \longrightarrow \text{Lindelöf}$$

Theorem (Todorčević 1995)

For general topological spaces, there are γ-sets whose product is not Lindelöf.

Corollary

For general topological spaces, Hurewicz, Scheepers and Menger are not productive.

Products

γ-property: each open ω-cover contains a γ-cover

$$\begin{array}{ccccccccc} & & \text{Hurewicz} & & \text{Scheepers} & & \text{Menger} & & \\ \gamma\text{-property} & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Omega) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \mathcal{O}) & \longrightarrow & \text{Lindelöf} \end{array}$$

Theorem (Todorčević 1995)

For general topological spaces, there are γ-sets whose product is not Lindelöf.

Corollary

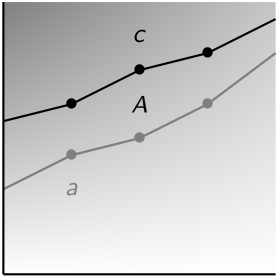
For general topological spaces, Hurewicz, Scheepers and Menger are not productive.

Problem (Open Problems in Topology II 2005)

Are there in ZFC Menger sets of reals whose product is not Menger?

Products

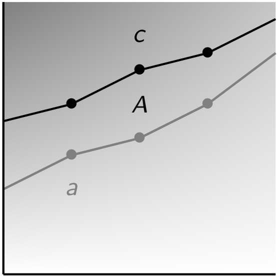
$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall\, c \in [\mathbb{N}]^\infty)(\,|\{\, a \in A : a \leq c \,\}| < \mathfrak{d}\,)$



Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall\, c \in [\mathbb{N}]^\infty)(\,|\{\, a \in A : a \leq c \,\}| < \mathfrak{d}\,)$

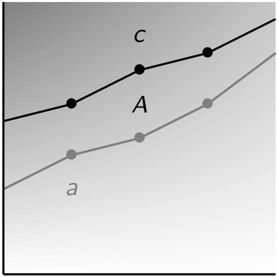
- A exists in ZFC



Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

- A exists in ZFC
- $A \cup \mathrm{Fin}$ is Menger



Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall\, c \in [\mathbb{N}]^\infty)(\,|\{\, a \in A : a \leq c\,\}| < \mathfrak{d}\,)$

Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded or $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded or $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d} \leq \mathfrak{r}$, there are Menger sets whose product is not Menger.

Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded or $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d} \leq \mathfrak{r}$, there are Menger sets whose product is not Menger.

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller

Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded or $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d} \leq \mathfrak{r}$, there are Menger sets whose product is not Menger.

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
✗	✗	✗	✗	✗	✗	✗	

Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded or $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d} \leq \mathfrak{r}$, there are Menger sets whose product is not Menger.

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
✗	✗	✗	✗	✗	✗	✗	✓

Theorem (Zdomskyy 2018)

In the Miller model (where $\mathfrak{d} > \mathfrak{r}$), Menger is productive.

Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded or $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d} \leq \mathfrak{r}$, there are Menger sets whose product is not Menger.

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
✗	✗	✗	✗	✗	✗	✗	✓

Theorem (Zdomskyy 2018)

In the Miller model (where $\mathfrak{d} > \mathfrak{r}$), Menger is productive.

Hurewicz $\to$ U-Menger $\to$ Scheepers $\to$ Menger

Products

$[\mathbb{N}]^\infty \supseteq A$ is $\mathfrak{d}$-unbounded: $|A| \geq \mathfrak{d}$ and $(\forall c \in [\mathbb{N}]^\infty)(|\{a \in A : a \leq c\}| < \mathfrak{d})$

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded or $\mathrm{cf}(\mathfrak{d})$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{d} \leq \mathfrak{r}$, there are Menger sets whose product is not Menger.

MA	Cohen	Random	Sacks	Hechler	Laver	Mathias	Miller
✗	✗	✗	✗	✗	✗	✗	✓

Theorem (Zdomskyy 2018)

In the Miller model (where $\mathfrak{d} > \mathfrak{r}$), Menger is productive.

Corollary

In the Miller model, each Hurewicz set is productively Menger.

Products

Theorem (Miller–Tsaban–Zdomskyy 2016)

Under CH, there are γ-sets X and Y whose product $X \times Y$ is not Menger.

Products

Theorem (Miller–Tsaban–Zdomskyy 2016)

Under CH, there are γ-sets X and Y whose product $X \times Y$ is not Menger.

Theorem (Repovš–Zdomskyy 2019)

In the Miller model, each ground model γ-set is a γ-set in the extension.

Products

Theorem (Miller–Tsaban–Zdomskyy 2016)

Under CH, there are γ-sets X and Y whose product $X \times Y$ is not Menger.

Theorem (Repovš–Zdomskyy 2019)

In the Miller model, each ground model γ-set is a γ-set in the extension.

Theorem (Miller 1984)

In the Miller model, each ground model non-Menger set is non-Hurewicz in the extension.

Products

Theorem (Miller–Tsaban–Zdomskyy 2016)

Under CH, there are γ-sets X and Y whose product $X \times Y$ is not Menger.

Theorem (Repovš–Zdomskyy 2019)

In the Miller model, each ground model γ-set is a γ-set in the extension.

Theorem (Miller 1984)

In the Miller model, each ground model non-Menger set is non-Hurewicz in the extension.

Corollary

In the Miller model, Hurewicz is not productive.

Products

Theorem (Miller–Tsaban–Zdomskyy 2016)

Under CH, there are γ-sets X and Y whose product $X \times Y$ is not Menger.

Theorem (Repovš–Zdomskyy 2019)

In the Miller model, each ground model γ-set is a γ-set in the extension.

Theorem (Miller 1984)

In the Miller model, each ground model non-Menger set is non-Hurewicz in the extension.

Corollary

In the Miller model, Hurewicz is not productive.

Problem

Is it true that, in the Miller model, any Sierpiński set from the ground model is Hurewicz in the extension?

Productive sets

Fact

Each σ-compact set is productively Hurewicz (Scheepers, Menger).

Productive sets

Fact

Each σ-compact set is productively Hurewicz (Scheepers, Menger).

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b} \}$ is a $\mathfrak{b}$-scale: unbounded and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$

Productive sets

Fact

Each σ-compact set is productively Hurewicz (Scheepers, Menger).

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b} \}$ is a $\mathfrak{b}$-scale: unbounded and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$

Theorem (Miller–Tsaban–Zdomskyy 2014)

If $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $X \cup \mathrm{Fin}$ is productively Hurewicz.

Productive sets

Fact

Each σ-compact set is productively Hurewicz (Scheepers, Menger).

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b} \}$ is a $\mathfrak{b}$-scale: unbounded and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$

Theorem (Miller–Tsaban–Zdomskyy 2014)

If $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $X \cup \mathrm{Fin}$ is productively Hurewicz.

Theorem (Repovš–Zdomskyy 2024)

It is consistent with CH that there is a $\mathfrak{b}$-scale $X \subseteq [\mathbb{N}]^\infty$ and a Menger set Y such that $(X \cup \mathrm{Fin}) \times Y$ is not Menger.

Productive sets

Fact

Each σ-compact set is productively Hurewicz (Scheepers, Menger).

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b} \}$ is a $\mathfrak{b}$-scale: unbounded and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$

Theorem (Miller–Tsaban–Zdomskyy 2014)

If $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $X \cup \mathrm{Fin}$ is productively Hurewicz.

Theorem (Repovš–Zdomskyy 2024)

It is consistent with CH that there is a $\mathfrak{b}$-scale $X \subseteq [\mathbb{N}]^\infty$ and a Menger set Y such that $(X \cup \mathrm{Fin}) \times Y$ is not Menger.

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b} \}$ is a scale: dominating and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$

Productive sets

Fact

Each σ-compact set is productively Hurewicz (Scheepers, Menger).

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b} \}$ is a $\mathfrak{b}$-scale: unbounded and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$

Theorem (Miller–Tsaban–Zdomskyy 2014)

If $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $X \cup \mathrm{Fin}$ is productively Hurewicz.

Theorem (Repovš–Zdomskyy 2024)

It is consistent with CH that there is a $\mathfrak{b}$-scale $X \subseteq [\mathbb{N}]^\infty$ and a Menger set Y such that $(X \cup \mathrm{Fin}) \times Y$ is not Menger.

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b} \}$ is a scale: dominating and $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$

Theorem (Miller–Tsaban–Zdomskyy 2014)

If $X \subseteq [\mathbb{N}]^\infty$ is a scale, then $X \cup \mathrm{Fin}$ is productively Hurewicz (Scheepers, Menger)

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

For $x, y \in [\mathbb{N}]^\infty$, $x \leq_U y$ if $\{\, n : x(n) \leq y(n) \,\} \in U$

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

For $x, y \in [\mathbb{N}]^\infty$, $x \leq_U y$ if $\{ n : x(n) \leq y(n) \} \in U$

$\mathfrak{b}(U)$: minimal cardinality of a $\leq_U$-unbounded set

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

For $x, y \in [\mathbb{N}]^\infty$, $x \leq_U y$ if $\{\, n : x(n) \leq y(n) \,\} \in U$

$\mathfrak{b}(U)$: minimal cardinality of a $\leq_U$-unbounded set

$[\mathbb{N}]^\infty \supseteq \{\, x_\alpha : \alpha < \mathfrak{b}(U) \,\}$ is a U-scale if it is $\leq_U$-unbounded and

$$x_\alpha \leq_U x_\beta \text{ for } \alpha < \beta$$

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

For $x, y \in [\mathbb{N}]^\infty$, $x \leq_U y$ if $\{ n : x(n) \leq y(n) \} \in U$

$\mathfrak{b}(U)$: minimal cardinality of a $\leq_U$-unbounded set

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b}(U) \}$ is a U-scale if it is $\leq_U$-unbounded and

$$x_\alpha \leq_U x_\beta \text{ for } \alpha < \beta$$

U-Menger:
$(\forall\, \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\forall\, x)(\{ n : x \in \bigcup \mathcal{F}_n \} \in U)$

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

For $x, y \in [\mathbb{N}]^\infty$, $x \leq_U y$ if $\{\, n : x(n) \leq y(n) \,\} \in U$

$\mathfrak{b}(U)$: minimal cardinality of a $\leq_U$-unbounded set

$[\mathbb{N}]^\infty \supseteq \{\, x_\alpha : \alpha < \mathfrak{b}(U) \,\}$ is a U-scale if it is $\leq_U$-unbounded and

$$x_\alpha \leq_U x_\beta \text{ for } \alpha < \beta$$

U-Menger:
$(\,\forall\, \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O}\,)(\,\exists\, \text{fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots\,)(\,\forall\, x\,)(\,\{\, n : x \in \bigcup \mathcal{F}_n \,\} \in U\,)$

Hurewicz $\to$ U-Menger $\to$ Scheepers $\to$ Menger

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

For $x, y \in [\mathbb{N}]^\infty$, $x \leq_U y$ if $\{\, n : x(n) \leq y(n) \,\} \in U$

$\mathfrak{b}(U)$: minimal cardinality of a $\leq_U$-unbounded set

$[\mathbb{N}]^\infty \supseteq \{\, x_\alpha : \alpha < \mathfrak{b}(U) \,\}$ is a U-scale if it is $\leq_U$-unbounded and

$$x_\alpha \leq_U x_\beta \text{ for } \alpha < \beta$$

U-Menger:
$(\forall\, \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\forall\, x)(\{\, n : x \in \bigcup \mathcal{F}_n \,\} \in U)$

Hurewicz $\to$ U-Menger $\to$ Scheepers $\to$ Menger

Lemma (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ is a U-scale, then $X \cup \mathrm{Fin}$ is productively U-Menger.

U-Menger and products

$[\mathbb{N}]^\infty \supseteq U$: nonprincipal ultrafilter

For $x, y \in [\mathbb{N}]^\infty$, $x \leq_U y$ if $\{ n : x(n) \leq y(n) \} \in U$

$\mathfrak{b}(U)$: minimal cardinality of a $\leq_U$-unbounded set

$[\mathbb{N}]^\infty \supseteq \{ x_\alpha : \alpha < \mathfrak{b}(U) \}$ is a U-scale if it is $\leq_U$-unbounded and

$$x_\alpha \leq_U x_\beta \text{ for } \alpha < \beta$$

U-Menger:
$(\forall \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O})(\exists \text{ fin } \mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots)(\forall x)(\{ n : x \in \bigcup \mathcal{F}_n \} \in U)$

Hurewicz $\rightarrow$ U-Menger $\rightarrow$ Scheepers $\rightarrow$ Menger

Lemma (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ is a U-scale, then $X \cup \mathrm{Fin}$ is productively U-Menger.

Theorem (Sz, Tsaban, Zdomskyy 2021)

Assume that $\mathfrak{d} \leq \mathfrak{r}$ and $\mathfrak{d}$ is regular. Then there are sets Menger in all finite powers (and thus Scheepers) whose product is not Menger.

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ *contains a* $\mathfrak{d}$*-unbounded set, then there is a* $\mathfrak{d}$*-unbounded set* $Y \subseteq [\mathbb{N}]^\infty$ *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ *contains a* $\mathfrak{d}$*-unbounded set, then there is a* $\mathfrak{d}$*-unbounded set* $Y \subseteq [\mathbb{N}]^\infty$ *such that* $X \times (Y \cup \mathrm{Fin})$ *is not Menger.*

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$*, each productively Menger set is productively Hurewicz.*

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Menger set is productively Hurewicz.

Theorem (Repovš–Zdomskyy 2024)

In the Laver model, each Hurewicz set is productively Menger.

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Menger set is productively Hurewicz.

Theorem (Repovš–Zdomskyy 2024)

In the Laver model, each Hurewicz set is productively Menger.

Corollary

In the Laver model (where $\mathfrak{b} = \mathfrak{d}$),

$$\begin{tikzcd} \textit{productively Menger} \arrow[r] & \textit{productively Hurewicz} \arrow[r] & \textit{Hurewicz} \arrow[ll, bend right] \end{tikzcd}$$

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Menger set is productively Hurewicz.

Theorem (Repovš–Zdomskyy 2024)

In the Laver model, each Hurewicz set is productively Menger.

Corollary

In the Laver model (where $\mathfrak{b} = \mathfrak{d}$),

$$\begin{tikzcd}
\textit{productively Menger} \arrow[r] & \textit{productively Hurewicz} \arrow[r] & \textit{Hurewicz} \arrow[ll, bend right]
\end{tikzcd}$$

Hurewicz $\to$ Scheepers $\to$ Menger

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Menger set is productively Hurewicz.

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Menger set is productively Hurewicz.

Theorem (Sz–Zdomskyy 2024)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set and $\mathfrak{d}$ is regular, then there is an ultrafilter U and a U-scale $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Menger set is productively Hurewicz.

Theorem (Sz–Zdomskyy 2024)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set and $\mathfrak{d}$ is regular, then there is an ultrafilter U and a U-scale $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Zdomskyy 2024)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Scheepers set is productively Hurewicz.

Products in the Laver model

Theorem (Sz–Tsaban 2017)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set, then there is a $\mathfrak{d}$-unbounded set $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Tsaban 2017)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Menger set is productively Hurewicz.

Theorem (Sz–Zdomskyy 2024)

If $X \subseteq [\mathbb{N}]^\infty$ contains a $\mathfrak{d}$-unbounded set and $\mathfrak{d}$ is regular, then there is an ultrafilter U and a U-scale $Y \subseteq [\mathbb{N}]^\infty$ such that $X \times (Y \cup \mathrm{Fin})$ is not Menger.

Theorem (Sz–Zdomskyy 2024)

Assuming $\mathfrak{b} = \mathfrak{d}$, each productively Scheepers set is productively Hurewicz.

Hurewicz $\rightarrow$ Scheepers $\rightarrow$ Menger

Products in the Laver model

$\mathcal{Q}[X]$: all increasing sequences of countable compact sets $Q = \langle K_n^Q : n \in \omega \rangle$

Lemma (Sz–Zdomskyy 2024)

In the Laver model, a set X is Hurewicz iff for each function $f : \mathcal{Q}[X] \to \mathbb{N}^{\mathbb{N}}$, there is a family $\mathcal{Q}_1 \subseteq \mathcal{Q}[X]$ of size ω_1 such that for each finite set $F \subseteq X$, there is $Q \in \mathcal{Q}_1$ such that $F \subseteq B(K_n^Q, \frac{1}{f_Q(n)})$ for all but finitely many n.

Products in the Laver model

$\mathcal{Q}[X]$: all increasing sequences of countable compact sets $Q = \langle\, K_n^Q : n \in \omega \,\rangle$

Lemma (Sz–Zdomskyy 2024)

In the Laver model, a set X is Hurewicz iff for each function $f : \mathcal{Q}[X] \to \mathbb{N}^{\mathbb{N}}$, there is a family $\mathcal{Q}_1 \subseteq \mathcal{Q}[X]$ of size ω_1 such that for each finite set $F \subseteq X$, there is $Q \in \mathcal{Q}_1$ such that $F \subseteq B(K_n^Q, \frac{1}{f_Q(n)})$ for all but finitely many n.

Theorem (Sz–Zdomskyy 2024)

In the Laver model, the following assertions are equivalent

1. *X is Hurewicz*
2. *X satisfies the property from Lemma*
3. *X is productively Menger*
4. *X is productively Scheepers*
5. *X is productively Hurewicz*

Products in the Laver model

$\mathcal{Q}[X]$: all increasing sequences of countable compact sets $Q = \langle K_n^Q : n \in \omega \rangle$

Lemma (Sz–Zdomskyy 2024)

In the Laver model, a set X is Hurewicz iff for each function $f : \mathcal{Q}[X] \to \mathbb{N}^{\mathbb{N}}$, there is a family $\mathcal{Q}_1 \subseteq \mathcal{Q}[X]$ of size ω_1 such that for each finite set $F \subseteq X$, there is $Q \in \mathcal{Q}_1$ such that $F \subseteq B(K_n^Q, \frac{1}{f_Q(n)})$ for all but finitely many n.

Theorem (Sz–Zdomskyy 2024)

In the Laver model, the following assertions are equivalent

1. *X is Hurewicz*
2. *X satisfies the property from Lemma*
3. *X is productively Menger*
4. *X is productively Scheepers*
5. *X is productively Hurewicz*

Corollary (from one of the previous results)

In the Laver model, Scheepers and Menger are not productive.

General spaces

$$\begin{array}{ccccccccc} & & \text{Hurewicz} & & \text{Scheepers} & & \text{Menger} & & \\ \gamma\text{-property} & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) & \longrightarrow & \text{Lindelöf} \end{array}$$

Theorem (Todorčević 1995)

For general topological spaces, there are γ-sets whose product is not Lindelöf.

General spaces

$$\begin{array}{ccccccccc} & & \text{Hurewicz} & & \text{Scheepers} & & \text{Menger} & & \\ \gamma\text{-property} & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) & \longrightarrow & \text{Lindelöf} \end{array}$$

Theorem (Todorčević 1995)

For general topological spaces, there are γ-sets whose product is not Lindelöf.

Theorem (Zdomskyy 2016)

In the Miller model, if a product of Menger spaces is Lindelöff, then it is Menger.

General spaces

$$\begin{array}{ccccccccc} & & \text{Hurewicz} & & \text{Scheepers} & & \text{Menger} & & \\ \gamma\text{-property} & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Omega) & \longrightarrow & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) & \longrightarrow & \text{Lindelöf} \end{array}$$

Theorem (Todorčević 1995)

For general topological spaces, there are γ-sets whose product is not Lindelöf.

Theorem (Zdomskyy 2016)

In the Miller model, if a product of Menger spaces is Lindelöff, then it is Menger.

Theorem (Sz–Zdomskyy 2024)

In the Laver model, if X is Hurewicz, Y is Hurewicz (Scheepers, Menger) and the product $X \times Y$ is Lindelöff, then $X \times Y$ is Hurewicz (Scheepers, Menger).

General spaces

Theorem (Michael 1971)

Under CH, there is an uncountable subset of the Sorgenfrey line whose all finite powers are Lindelöff.

General spaces

Theorem (Michael 1971)

Under CH, there is an uncountable subset of the Sorgenfrey line whose all finite powers are Lindelöff.

Theorem (Sz–Tsaban 2019)

Assuming $\mathfrak{b} = \omega_1$, for general topological spaces, if $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $(X \cup \mathrm{Fin})_M$ with the Michael topology is productively Hurewicz. In particular $(X \cup \mathrm{Fin})_M$ is Hurewicz (and thus Lindelöff) in all finite powers.

General spaces

Theorem (Michael 1971)

Under CH, there is an uncountable subset of the Sorgenfrey line whose all finite powers are Lindelöff.

Theorem (Sz–Tsaban 2019)

Assuming $\mathfrak{b} = \omega_1$, for general topological spaces, if $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $(X \cup \mathrm{Fin})_M$ with the Michael topology is productively Hurewicz. In particular $(X \cup \mathrm{Fin})_M$ is Hurewicz (and thus Lindelöff) in all finite powers.

$$\mathrm{id}\colon (X \cup \mathrm{Fin})_M \to (X \cup \mathrm{Fin})_S$$

General spaces

Theorem (Michael 1971)

Under CH, there is an uncountable subset of the Sorgenfrey line whose all finite powers are Lindelöff.

Theorem (Sz–Tsaban 2019)

Assuming $\mathfrak{b} = \omega_1$, for general topological spaces, if $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $(X \cup \mathrm{Fin})_M$ with the Michael topology is productively Hurewicz. In particular $(X \cup \mathrm{Fin})_M$ is Hurewicz (and thus Lindelöff) in all finite powers.

$$\mathrm{id}\colon (X \cup \mathrm{Fin})_M \to (X \cup \mathrm{Fin})_S$$

Corollary

Assuming $\mathfrak{b} = \omega_1$, if $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $(X \cup \mathrm{Fin})_S$ is Hurewicz (and thus Lindelöff) in all finite powers.

General spaces

Theorem (Michael 1971)

Under CH, there is an uncountable subset of the Sorgenfrey line whose all finite powers are Lindelöff.

Theorem (Sz–Tsaban 2019)

Assuming $\mathfrak{b} = \omega_1$, for general topological spaces, if $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $(X \cup \mathrm{Fin})_M$ with the Michael topology is productively Hurewicz. In particular $(X \cup \mathrm{Fin})_M$ is Hurewicz (and thus Lindelöff) in all finite powers.

$$\mathrm{id}\colon (X \cup \mathrm{Fin})_M \to (X \cup \mathrm{Fin})_S$$

Corollary

Assuming $\mathfrak{b} = \omega_1$, if $X \subseteq [\mathbb{N}]^\infty$ is a $\mathfrak{b}$-scale, then $(X \cup \mathrm{Fin})_S$ is Hurewicz (and thus Lindelöff) in all finite powers.

Theorem (Todorčević 1989)

Under OCA, for each uncountable subset X of the Sorgenfrey line, X^2 contains a closed discrete uncountable subspace.

General spaces

Theorem (Sz–Tsaban 2019)

Assume that $\mathfrak{d} = \omega_1$. *Each productively Lindelöf space is productively Menger and each productively Menger space is productively Hurewicz.*

General spaces

Theorem (Sz–Tsaban 2019)

Assume that $\mathfrak{d} = \omega_1$. *Each productively Lindelöf space is productively Menger and each productively Menger space is productively Hurewicz.*

Theorem (Sz–Tsaban 2019)

If $X \subseteq [\mathbb{N}]^\infty$ *is a scale of size* ω_1, *then* $X \cup \mathrm{Fin}$ *with the Michael topology is productively Menger but no productively Lindelöf.*

General spaces

Theorem (Sz–Tsaban 2019)

Assume that $\mathfrak{d} = \omega_1$. Each productively Lindelöf space is productively Menger and each productively Menger space is productively Hurewicz.

Theorem (Sz–Tsaban 2019)

If $X \subseteq [\mathbb{N}]^\infty$ is a scale of size ω_1, then $X \cup \mathrm{Fin}$ with the Michael topology is productively Menger but no productively Lindelöf.

Theorem (Blass, Sz–Tsaban 2019)

It is consistent with CH that there is a productively Hurewicz space which is not productively Menger.

General spaces

Theorem (Sz–Tsaban 2019)

Assume that $\mathfrak{d} = \omega_1$. Each productively Lindelöf space is productively Menger and each productively Menger space is productively Hurewicz.

Theorem (Sz–Tsaban 2019)

If $X \subseteq [\mathbb{N}]^\infty$ is a scale of size ω_1, then $X \cup \mathrm{Fin}$ with the Michael topology is productively Menger but no productively Lindelöf.

Theorem (Blass, Sz–Tsaban 2019)

It is consistent with CH that there is a productively Hurewicz space which is not productively Menger.

Theorem (Repovš–Zdomskyy 2024)

It is consistent with CH that there is a $\mathfrak{b}$-scale $X \subseteq [\mathbb{N}]^\infty$ and a Menger set Y such that $(X \cup \mathrm{Fin}) \times Y$ is not Menger.