Fractal dimension and common hypercyclicity

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38th Summer conference on topology and its applications

July 9, 2024

Linear dynamical systems

Definition : Linear dynamical system

A **linear dynamical system** is a pair (X, T) consisting of a Banach space (or Fréchet space) X and a continuous linear operator $T : X \to X$.

Example

The pair (ℓ^p, B) where B is given by

$$B(x_0, x_1, x_2, x_3, \cdots) = (x_1, x_2, x_3, x_4, \cdots)$$

is a linear dynamical system.

Hypercyclicity

Definition : Hypercyclicity

Let (X, T) be a linear dynamical system. The operator T is said to be **hypercyclic** if there exists a vector $x \in X$ (said to be hypercyclic) such that

$$\mathrm{Orb}(x, T) := \{T^n x : n \geq 0\} \quad \text{is dense in } X.$$

Moreover, we denote by $\mathrm{HC}(T)$ the set of hypercyclic vectors of T.

Theorem [Rolewicz 1969]

There is no hypercyclic operator on any finite-dimensional Banach space.

Theorem [Ansari '97, Bernal '99, Bonet-Peris '98]

Every separable infinite-dimensional Fréchet space supports a hypercyclic operator.

Examples of hypercyclic operators

- Birkhoff (1929) : the translation operators T_a on $H(\mathbb{C})$ defined by

$$T_a f(z) = f(z+a), \ a \in \mathbb{C}\backslash\{0\}.$$

- MacLane (1952) : the derivative operator D on $H(\mathbb{C})$ defined by

$$Df = f'.$$

- Rolewicz (1969) : the multiples of backward shift λB on ℓ^p defined by

$$\lambda B(x_0, x_1, x_2, x_3, \cdots) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \cdots), \ |\lambda| > 1.$$

How to prove that an operator is hypercyclic ?

Birkhoff transitivity theorem (1920)

Let (X, T) be a linear dynamical system. If X is separable then the following assertions are equivalent :

1. T is hypercyclic ;
2. T is topologically transitive, *i.e.* for every non-empty open sets U, V in X, there exists n such that
$$T^n U \cap V \neq \emptyset.$$

Moreover, if T is hypercyclic then $HC(T)$ is a dense G_δ-set.

Idea of the proof : Let $(U_k)_{k\geq 0}$ be a countable open basis of X. We have

$$HC(T) = \bigcap_{k\geq 1} \bigcup_{n\geq 0} T^{-n}(U_k).$$

How to prove that an operator is hypercyclic ?

Hypercyclicity Criterion [Kitai '82, Gethner-Shapiro '87, Bès-Peris '99]

Let X be a separable Fréchet space and $T \in L(X)$. If there exist a strictly increasing sequence $(n_k) \subset \mathbb{N}$, a dense set X_0 in X and maps $S_{n_k} : X_0 \to X_0$ such that for any $x \in X_0$,

$$T^{n_k} x \to 0, \quad S_{n_k} x \to 0 \quad \text{and} \quad T^{n_k} S_{n_k} x \to x \text{ as } k \text{ tends to } \infty,$$

then T is **hypercyclic**.

Idea of the proof : Let $(y_j)_{j \geq 0}$ be a dense sequence in X_0. We can show that if (n_{k_l}) grows sufficiently rapidly then the vector

$$x := y_0 + \sum_{l=1}^{\infty} S_{n_{k_l}} y_l$$

is well-defined and has a dense orbit.

Hypercyclicity of λB with $|\lambda| > 1$

1. **Birkhoff transitivity theorem** : Let U, V be non-empty open subsets of ℓ^p. Let $y = (y_0, \cdots, y_N, 0, \cdots) \in V$ and $x = (x_0, \cdots, x_N, 0, \cdots) \in U$. We can find $n > N$ such that

$$z = (x_0, \cdots, x_N, 0, \cdots, 0, \frac{y_0}{\lambda^n}, \cdots, \frac{y_N}{\lambda^n}, 0, \cdots) \in U$$

since $|\lambda| > 1$ and we have $(\lambda B)^n z = y \in V$.

2. **Hypercyclicity Criterion** : We consider $X_0 = c_{00}$, $n_k = k$ and $S_k x = \frac{1}{\lambda^k} F^k x$ where

$$F(x_0, x_1, x_2, x_3, \cdots) = (0, x_0, x_1, x_2, x_3, \cdots).$$

Therefore we remark that for every $x \in X_0$, we have

$$(\lambda B)^k x \to 0, \quad S_k x \to 0 \quad \text{and} \quad (\lambda B)^k S_k x \to x \text{ as } k \text{ tends to } \infty.$$

Hypercyclicity of direct sums of λB

Proposition

Let $X_1, \dots, X_d$ be separable Fréchet spaces and $T_i \in L(X_i)$ for each $1 \le i \le d$. If there exists a sequence (n_k) such that each operator T_i satisfies the Hypercyclicity Criterion along (n_k) then $T_1 \oplus \cdots \oplus T_d$ is hypercyclic on $X_1 \oplus \cdots \oplus X_d$.

Consequences : Let $\lambda \in]1, \infty[^d$. The operator $\lambda B := \lambda_1 B \oplus \cdots \oplus \lambda_d B$ is hypercyclic on $\ell^p \oplus \cdots \oplus \ell^p$.

Question :

Given $\Lambda \subset]1, \infty[^d$, does the family $(\lambda B)_{\lambda \in \Lambda}$ share a common hypercyclic vector, *i.e* $\bigcap_{\lambda \in \Lambda} HC(\lambda B) \neq \emptyset$?

Common hypercyclicity

Proposition

Let X be a separable Fréchet space and $(T_\lambda)_{\lambda\in\Lambda} \subset L(X)$ a family of hypercyclic operators. If Λ is countable then the family T_λ shares a common hypercyclic vector.

Proof : Direct consequence of Birkhoff transitivity theorem.

Consequence : If $\Lambda \subset]1,\infty[^d$ is a countable set then the family $(\lambda B)_{\lambda\in\Lambda}$ shares a common hypercyclic vector.

Theorem (Abakumov-Gordon 2003)

The family $(\lambda B)_{\lambda\in]1,\infty[}$ shares a common hypercyclic vector.

Idea of the proof : We extend the construction used in the proof of the Hypercyclicity Criterion. Given $1 < a < b < \infty$, $y \in c_{00}\setminus\{0\}$, $\varepsilon > 0$, we show that there exist N arbitrarily big and M such that if we let

$$x := \sum_{l=1}^{d} \frac{F^{N+lM} y}{\lambda_l^{N+lM}}$$

for some convenient choice of $\lambda_1, \dots, \lambda_d$ then for every $\lambda \in [a, b]$ there exists $1 \leq l \leq d$ such that

$$\|(\lambda B)^{N+lM} x - y\| < \varepsilon.$$

Good choice of λ_l : $\lambda_1 = a + \frac{\varepsilon}{2\|y\|}\frac{1}{N+M}$ and $\lambda_{l+1} = \lambda_l + \frac{\varepsilon}{2\|y\|}\frac{1}{N+(l+1)M}$.

Theorem (Abakumov-Gordon 2003)

If $d \geq 2$, the family $(\lambda B)_{\lambda \in]1,\infty[^d}$ does not share a common hypercyclic vector.

Idea of the proof : Let $\Lambda \subset]1,\infty[^d$. We can show that if $(\lambda B)_{\lambda \in \Lambda}$ shares a common hypercyclic vector then $\mathrm{Leb}(\Lambda) = 0$. Indeed, if x is a common hypercyclic vector for $(\lambda B)_{\lambda \in \Lambda}$ then ,

$$\Lambda \subset \bigcap_{N \geq 1} \bigcup_{n \geq N} \{\lambda \in]1,\infty[^d : \|(\lambda B)^n x - (e_0, \cdots, e_0)\| < \frac{1}{2}\}.$$

However, there exists a constant C such that for any n

$$\mathrm{Leb}\left(\{\lambda \in]1,\infty[^d : \|(\lambda B)^n x - (e_0, \cdots, e_0)\| < \frac{1}{2}\}\right) \leq \frac{C}{n^d}.$$

Since $d \geq 2$, it implies that $\mathrm{Leb}(\Lambda) = 0$.

A (technical) characterization

Theorem (Bayart, Costa Jr, M., 2022)

Let $\Lambda \subset (0,+\infty)^d$ be σ-compact, $X = \ell_p(\mathbb{N})$, $p \in [1,+\infty)$ or $X = c_0(\mathbb{N})$. The following assertions are equivalent :

1. $(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B)_{\lambda\in\Lambda}$ shares a common hypercyclic vector.
2. For all $\tau > 0$, for all $N \geq 1$, for all $K \subset \Lambda$ compact, there exist $N \leq n_1 < n_1 + N \leq n_2 < \cdots < n_{q-1} + N \leq n_q$ and $(\lambda_k)_{k=1,\dots,q} \in (0,+\infty)^d$ such that
   - $K \subset \cup_{k=1}^q \prod_{i=1}^d \left[\lambda_k(i) - \frac{\tau}{n_k}, \lambda_k(i)\right]$
   - for all $k = 1,\dots,q-1$, for all $i = 1,\dots,d$,

$$\lambda_{k+1}(i)n_{k+1} - \lambda_k(i)n_k \geq N.$$

Main Results I

Theorem (Bayart, Matheron 2007)

If $\Lambda \subset (0, +\infty)^d$ is a monotonic Lipschitz curve then $(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B)_{\lambda \in \Lambda}$ shares a common hypercyclic vector.

Theorem (Bayart, Costa Jr, M. 2022)

If $\Lambda \subset (0, +\infty)^d$ is a Lipschitz curve then $(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B)_{\lambda \in \Lambda}$ shares a common hypercyclic vector.

Main Results II

We recall that the Hausdorff dimension of a set Λ is given by

$$\dim_{\mathcal{H}}(\Lambda) = \inf\{s > 0 : \mathcal{H}^s(\Lambda) = 0\}$$

where

$$\mathcal{H}^s(\Lambda) = \lim_{\varepsilon \to 0} \inf_{\mathrm{diam}(A_i) < \varepsilon} \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(A_i)^s \right\}.$$

Theorem (Bayart, Costa Jr, M. 2022)

If $\Lambda \subset (0, +\infty)^d$ satisfies $\dim_{\mathcal{H}}(\Lambda) > 1$ then $(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B)_{\lambda \in \Lambda}$ does not admit a common hypercyclic vector.

Consequence : There exists $\Lambda \subset (0, +\infty)^2$ with $\mathrm{Leb}(\Lambda) = 0$ and $(e^{\lambda(1)}B \times \cdots \times e^{\lambda(d)}B)_{\lambda \in \Lambda}$ has no common hypercyclic vector.

Main Results III

Definition : Homogeneous box dimension

Let $\Lambda \subset \mathbb{R}^d$ be compact. We say that Λ has *homogeneous box dimension at most* $\gamma \in (0, d]$ if there exist $r \geq 2$, $C(\Lambda) > 0$ and, for all $m \geq 1$, a family $(\Lambda_{\mathbf{k}})_{\mathbf{k} \in \{1,\dots,r\}^m}$ of compact subsets of Λ such that for all $m \geq 1$,

- for all $\mathbf{k} \in \{1, \dots, r\}^m$, $\operatorname{diam}(\Lambda_{\mathbf{k}}) \leq C(\Lambda) \left(\frac{1}{r^{1/\gamma}}\right)^m$;
- $\Lambda \subset \bigcup_{\mathbf{k} \in \{1,\dots,r\}^m} \Lambda_{\mathbf{k}}$;
- for all $\mathbf{k} \in \{1, \dots, r\}^m$, $\Lambda_{k_1,\dots,k_m} \subset \Lambda_{k_1,\dots,k_{m-1}}$.

The *homogeneous box dimension* of Λ is defined as the infimum of the $\gamma \in (0, d]$ such that Λ has homogeneous box dimension at most γ and will be denoted $\dim_{HB}(\Lambda)$.

Theorem (Bayart, Costa Jr, M. 2022)

If $\Lambda \subset (0, +\infty)^d$ satisfies $\dim_{HB}(\Lambda) < 1$ then $(e^{\lambda(1)} B \times \cdots \times e^{\lambda(d)} B)_{\lambda \in \Lambda}$ admits a common hypercyclic vector.

Main Results IV

For every compact $\Lambda \subset \mathbb{R}^d$,

$$\dim_{\mathcal{H}}(\Lambda) \leq \dim_{HB}(\Lambda).$$

If Λ is selfsimilar with respect to r similarities with ratio ρ then

$$\dim_{\mathcal{H}}(\Lambda) = \dim_{HB}(\Lambda) = \frac{-\log(r)}{\log(\rho)}.$$

In this context, it remains the case $\dim_{\mathcal{H}}(\Lambda) = 1$!

This case has begun to be studied by Costa Jr (2024) that gives a positive answer under some additional connectedness assumptions on Λ and a study of the non-connected case is in progress.

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This case has begun to be studied by Costa Jr (2024) that gives a positive answer under some additional connectedness assumptions on Λ and a study of the non-connected case is in progress.
Warning : There exists a set $\Lambda \subset (0, +\infty)^2$ with $\dim_{\mathcal{H}}(\Lambda) = 1$ such that $(e^{\lambda(1)}B \times e^{\lambda(2)}B)_{\lambda \in \Lambda}$ does not admit a common hypercyclic vector.

Thank you for your attention.