An introduction to Lebesgue integration in σ-frames

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Motivation

In *"Measure, randomness and sublocales"* (2012), Alex Simpson proved that:

Any measure on a fitted σ-frame L can be extended to a measure on the coframe $\mathsf{S}(L)$ of all σ-sublocales.

Our goal:

- To describe the integral with respect to a measure defined on $\mathsf{S}(L)$;
- To extend the notion of integrable function to localic general functions.

Background

σ-Frames e σ-locales

A *σ-frame* L is a lattice with countable joins satisfying the distributive law

$$b \wedge \bigvee_{a \in A} a = \bigvee_{a \in A} b \wedge a$$

for every countable $A \subseteq L$ and $b \in L$.

A *σ-frame homomorphism* is a map $f : L \to M$ between σ-frames that preserves countable joins and finite meets.

- The category of *σ-locales* and *σ-localic maps* is the opposite category of the category of σ-frames and σ-frames homomorphisms:

$$\sigma\mathsf{Loc} = \sigma\mathsf{Frm}^{op}.$$

σ-sublocales

- *σ-sublocales* of a σ-locale L are described as σ-frame quocients L/θ given by *σ-frame congruences* θ on L, that is, equivalence relations on L satisfying the congruence properties

$$(x,y),(x',y')\in\theta \;\Rightarrow\; (x\wedge x', y\wedge y')\in\theta,$$

$$(x_a,y_a)\in\theta\ (a\in A, A=\text{countable set}) \;\Rightarrow\; \Big(\bigvee_{a\in A} x_a, \bigvee_{a\in A} y_a\Big)\in\theta.$$

Thus:

$$\mathsf{S}(L)=\mathcal{C}(L)^{op},$$

where $\mathcal{C}(L)$ denotes the frame of congruences on L and $\mathsf{S}(L)$ denotes the coframe of σ-sublocales.

Frame of reals

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all elements $(p, -)$ and $(-, q)$, with $p, q \in \mathbb{Q}$, and relations

(R_1) $(p, -) \wedge (-, q) = 0$ whenever $p \geq q$;

(R_2) $(p, -) \vee (-, q) = 1$ whenever $p < q$;

(R_3) $(p, -) = \bigvee\{(r, -) \mid p < r\}$;

(R_4) $(-, q) = \bigvee\{(-, s) \mid s < q\}$;

(R_5) $1 = \bigvee\{(p, -) \mid p \in \mathbb{Q}\}$;

(R_6) $1 = \bigvee\{(-, q) \mid q \in \mathbb{Q}\}$.

Remark : A σ-frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to L$ can be defined through a map from the generating set of $\mathfrak{L}(\mathbb{R})$ into a σ-frame L that sends the relations of $\mathfrak{L}(\mathbb{R})$ into identities in L.

Localic measurable functions

A *localic measurable real function on a σ-frame L* is a σ-frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to L$. We denote

$$\mathsf{M}(L) \coloneqq \sigma\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L).$$

A *localic real-valued function on a σ-frame L* is a σ-frame homomorphism $f : \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$. We denote

$$\mathsf{F}(L) \coloneqq \sigma\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{C}(L)) = \mathsf{M}(\mathcal{C}(L)).$$

Remark: As $\nabla : L \to \nabla[L]$[1] is an embedding of L in $\mathcal{C}(L)$,

$$\mathsf{M}(L) \subseteq \mathsf{F}(L).$$

[1] $\nabla[L] \coloneqq \{\nabla_a \mid a \in L\}$, where $\nabla_a \coloneqq \{(x, y) \mid x \vee a = y \vee a\}$, is the set of all closed congruences on L

Ring of localic measurable functions

- For any σ-frame L, we can endow $\mathsf{M}(L)$ with a partial order $\leq$ and operations of sum $(+)$, product $(\cdot)$ and scalar product, such that:
  - $\mathsf{M}(L)$ is an algebra over $\mathbb{Q}$;
  - $\mathsf{M}(L)$ is a lattice ordered commutative ring.

$*$ In particular, this holds for $\mathsf{M}(\mathcal{C}(L)) = \mathsf{F}(L)$.

Simple functions

Characteristic function

Let L be a σ-frame.

Recall: For any complemented $a \in L$, the measurable function

$$\chi_a \colon \mathfrak{L}(\mathbb{R}) \to L$$

determined by

$$\chi_a(p,-) = \begin{cases} 1 & \text{if } p < 0 \\ a & \text{if } 0 \le p < 1 \\ 0 & \text{if } p \ge 1 \end{cases} \quad \text{and} \quad \chi_a(-,q) = \begin{cases} 0 & \text{if } q \le 0 \\ a^c & \text{if } 0 < q \le 1 \\ 1 & \text{if } q > 1 \end{cases}$$

is the *characteristic function* [2] *associated with $a \in L$*.

[2] This is the point-free counterpart of the standard indicator (characteristic) function $\mathbb{1}_A : X \to \{0,1\}$, for a set X and an $A \subseteq X$.

Measurable simple functions

Definition: An $f \in \mathsf{M}(L)$ is a *measurable simple function* on L when

$$f = \sum_{i=1}^{n} r_i \cdot \chi_{a_i}$$

for some $n \in \mathbb{N}$, $r_1, \dots, r_n \in \mathbb{Q}$ and $a_1, \dots, a_n \in BL$ [3].

Remark: Whenever $r_1 < r_2 < \cdots < r_n$ and $a_1, \dots, a_n \in BL \setminus \{0\}$ are pairwise disjoint with $\bigvee_{i=1}^{n} a_i = 1$, we say that $\sum_{i=1}^{n} r_i \cdot \chi_{a_i}$ is the *canonical representation* of f.

A note on measure theory: given a measurable space $(X, \mathcal{A})$, a simple function is a map $f : X \to \mathbb{R}$ that is a linear combination of indicator functions associated with measurable sets.

[3] $BL := \{a \in L \mid a \text{ is complemented}\}$

Properties

Proposition: Any measurable simple function on L has one and only one canonical representation.

Set

$$\mathsf{SM}(L) := \{f \in \mathsf{M}(L) \mid f \text{ is simple}\}.$$

Proposition: $\mathsf{SM}(L)$ is a subring of $\mathsf{M}(L)$. In particular, this means that for any $f, g \in \mathsf{SM}(L) \subseteq \mathsf{M}(L)$ and $\lambda \in \mathbb{Q}$, $\lambda \cdot f$, $-f$, $f \cdot g$ and $f + g$ are simple measurable functions.

Integral of simple functions

Measure

Recall: A map $\mu\colon L \to [0,\infty]$ on a lattice L with countable joins is a *measure* on L if

(M1) $\mu(0_L) = 0$;

(M2) $\forall x, y \in L$, $x \leq y \Rightarrow \mu(x) \leq \mu(y)$;

(M3) $\forall x, y \in L$, $\mu(x) + \mu(y) = \mu(x \vee y) + \mu(x \wedge y)$;

(M4) $\forall (x_i)_{i\in\mathbb{N}}$ increasing in $L \Rightarrow \mu(\bigvee_{i\in\mathbb{N}} x_i) = \sup_{i\in\mathbb{N}} \mu(x_i)$.

From now on, let $\underline{L \text{ be a } \sigma\text{-frame}}$ and let $\underline{\mu \text{ be a measure on } \mathsf{S}(L)}$.

Goal

$\longrightarrow$ We have a measure μ on $\mathsf{S}(L)$.

Recalling that $\mathcal{C}(L) = \mathsf{S}(L)^{op}$, this suggests that we could try to define an integral for general localic real-valued functions

$$f : \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L) \in \mathsf{F}(L).$$

Notation: For each $S \in \mathsf{S}(L)$, we denote the corresponding congruence by θ_S (hence $S = L/\theta_S$). If S is complemented, S^c is the sigma-sublocale defined by θ_S^c.

Integral of a nonnegative simple function

Recall: Since $\mathsf{F}(L) = \mathsf{M}(\mathcal{C}(L))$, a real-valued function on L is simple if it is a measurable simple function on $\mathcal{C}(L)$.

Definition: If $g \in \mathsf{F}(L)$ is a nonnegative [4] simple function with canonical representation

$$g = \sum_{i=1}^{n} r_i \cdot \chi_{\theta^c_{S_i}},$$

the *μ-integral of g* is the value

$$\int_L g\, d\mu \equiv \int g\, d\mu := \sum_{i=1}^{n} r_i \mu(S_i) \quad \in [0, +\infty].$$

[4] An $f \in \mathsf{F}(L)$ is nonnegative if $f \geq \mathbf{0}$. The canonical representation of a nonnegative simple function has nonnegative scalars, i.e., $r_i \geq 0$ for $i = 1, ..., n$.

Integral of a nonnegative simple function

Definition (cont.): If $g \in \mathsf{F}(L)$ is a nonnegative simple function with canonical representation

$$g = \sum_{i=1}^{n} r_i \cdot \chi_{\theta^c_{S_i}},$$

for each $S \in \mathsf{S}(L)$, the *μ-integral of g over S* is given by

$$\int_S g \, d\mu := \sum_{i=1}^{n} r_i \mu(S_i \wedge S).$$

A note on measure theory: In a measure space, the integral of a simple function over a subset is restricted to the measurable subsets.

Generalising the integral to a general simple function

Given an $f \in \mathsf{F}(L)$, we define:

- the *positive part of f*: $f^+ := f \vee \mathbf{0}$;
- the *negative part of f*: $f^- := (-f) \vee \mathbf{0}$.

Moreover, for any $f \in \mathsf{F}(L)$, we have

$$f = f^+ - f^-.$$

THUS: The idea is to define the integral of a general simple function $g : \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ through the integrals of g^+ and g^-.

Integral of a general simple function

Definition: Let $S \in \mathsf{S}(L)$. A $g \in \mathsf{SM}(\mathcal{C}(L))$ is *μ-integrable over S* if

$$\int_S g^+ \, d\mu < \infty \qquad \text{or} \qquad \int_S g^- \, d\mu < \infty,$$

and the *μ-integral of g over S* is the value

$$\int_S g \, d\mu := \int_S g^+ \, d\mu - \int_S g^- \, d\mu \quad \in [0, +\infty].$$

We say that g is *μ-integrable* if g is μ-integrable over $L = 1_{\mathsf{S}(L)}$, and in that case we talk about the *μ-integral* of g.

Some properties

Proposition: If $g \in \mathsf{SM}(\mathcal{C}(L))$ is integrable over $S \in \mathcal{C}(L)$ and

$$g = \sum_{i=1}^{n} r_i \cdot \chi_{\theta^c_{S_i}}$$

is a representation of g with $\theta^c_{S_1}, \ldots, \theta^c_{S_n}$ pairwise disjoint in $B\mathcal{C}(L)$, then

$$\int_S g \, d\mu = \sum_{i=1}^{n} r_i \mu(S_i \wedge S).$$

Proposition: Let $g \in \mathsf{SM}(\mathcal{C}(L))$ and <u>let $S \in \mathsf{S}(L)$ be complemented</u>. If g is integrable over S, then $g \cdot \chi_{\theta^c_S}$ is integrable and

$$\int_S g \, d\mu = \int g \cdot \chi_{\theta^c_S} \, d\mu.$$

Some properties

Definition: A $g \in \mathsf{SM}(\mathcal{C}(L))$ is *summable over $S \in \mathsf{S}(L)$* if

$$\int_S g^+ \, d\mu < \infty \qquad \text{and} \qquad \int_S g^- \, d\mu < \infty.$$

We say that g is *summable* if g is summable over $L = 1_{\mathsf{S}(L)}$.

Proposition: The integral is linear on the class of summable simple functions, in the sense that for any $r, s \in \mathbb{Q}$ and any $g, h \in \mathsf{SM}(\mathcal{C}(L))$ summable over $S \in \mathsf{S}(L)$,

$$\int_S (r \cdot g + s \cdot h) \, d\mu = r \int_S g \, d\mu \, + \, s \int_S h \, \mu.$$

Some properties

Proposition: If $g \in \mathsf{SM}(\mathcal{C}(L))$ is integrable over a complemented σ-sublocale $S \in \mathsf{S}(L)$ and $\theta_S^c \wedge g(\text{—}, 0) = 0$ [5], then

$$\int_S g \, d\mu \geq 0.$$

Proposition: If $g, h \in \mathsf{SM}(\mathcal{C}(L))$ are integrable over a complemented $S \in \mathsf{S}(L)$ such that $\theta_S^c \wedge (h - g)(\text{—}, 0) = 0$, then

$$\int_S g \, d\mu \leq \int_S h \, d\mu.$$

[5] This condition can be roughly translated as "$g \geq \mathbf{0}$ in S".

The indefinite integral

Given a simple function $g \in \mathsf{F}(L)$, the map $\eta\colon \mathsf{S}(L) \to [0, \infty]$ defined by

$$\eta(S) \coloneqq \int_S g \, d\mu$$

is called the *indefinite integral* of g.

Proposition: The indefinite integral of a nonnegative simple function $f\colon \mathfrak{L}(\mathbb{R}) \to \mathcal{C}(L)$ is a measure on $\mathsf{S}(L)$.

Extending the integral to more general functions

Remark on limits

Given a σ-frame L, let

$$(f_k \colon \mathfrak{L}(\mathbb{R}) \to L)_{k \in \mathbb{N}}$$

be a sequence in $\mathsf{M}(L)$. Let us define:

- The *limit inferior* as $\lim_{k \to +\infty} \inf f_k \coloneqq \sup_{n \geq 1} \inf_{k \geq n} f_k$;
- The *limit superior* as $\lim_{k \to +\infty} \sup f_k \coloneqq \inf_{n \geq 1} \sup_{k \geq n} f_k$.

<u>The limit superior and the limit inferior may not exist.</u> When they both exist and are equal, we say that the *limit* of $(f_k)_{k \in \mathbb{N}}$ exists and write

$$\lim_{k \to +\infty} f_k = \lim_{k \to +\infty} \inf f_k = \lim_{k \to +\infty} \sup f_k.$$

Decomposing a nonnegative function

A NOTE ON MEASURE THEORY: Any nonnegative measurable function $f : X \to \mathbb{R}$ is a pointwise limit of an increasing sequence of nonnegative simple functions.

Proposition: Let $f \in \mathsf{F}(L)$ be a nonnegative real-valued function. If any countable join in $\{f(r, -) \mid r \in \mathbb{Q}\}$ is complemented in $\mathcal{C}(L)$, then there exists an increasing sequence $(g_k)_{k\in\mathbb{N}}$ in $\mathsf{SM}(\mathcal{C}(L))$ such that

$$\begin{cases} \mathbf{0} \leq g_k \leq f, \text{ for each } k \in \mathbb{N}, \\ f = \lim\limits_{k\to+\infty} g_k \end{cases}$$

Integral of a nonnegative function

Definition: Given a nonnegative $f \in \mathsf{F}(L)$, the *μ-integral of f over $S \in \mathsf{S}(L)$* is given by

$$\int_S f\, d\mu \coloneqq \sup\Big\{ \int_S g\, d\mu \mid \mathbf{0} \leq g \leq f,\ g \in \mathsf{SM}(\mathcal{C}(L)) \Big\}.$$

The μ-integral of f over $L = 1_{\mathsf{S}(L)}$ is called the *μ-integral of f*.

Integral of a general function

Definition: A function $f \in \overline{\mathsf{F}}(L)$ is *μ-integrable over $S \in \mathsf{S}(L)$* if

$$\int_S f^+ \, d\mu < \infty \qquad \text{or} \qquad \int_S f^- \, d\mu < \infty,$$

and its *μ-integral over S* is given by

$$\int_S f \, d\mu := \int_S f^+ \, d\mu - \int_S f^- \, d\mu.$$

The μ-integral of f over $L = 1_{\mathsf{S}(L)}$ is called the *μ-integral of f*.

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