Isometry Groups as Topological Groups

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Geometry

Geometry

- Riemannian Geometry
- Algebraic Geometry
- Metric Geometry

Def: Let (X,d) be a metric space

$$\mathrm{Iso}(X,d) :=$$

$$\{ f : X \to X \text{ bijective and } (\forall a,b \in X)\ d(f(a),f(b)) = d(a,b) \}$$

Isometry Group of (X,d)

$\mathrm{Iso}(X,d)$ is a topological group

Metric Spaces $\xrightarrow{\text{Iso}}$ Topological Groups

injective? No!

Surjective?

Is every topological group

the isometry group

of some metric space?

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Example:

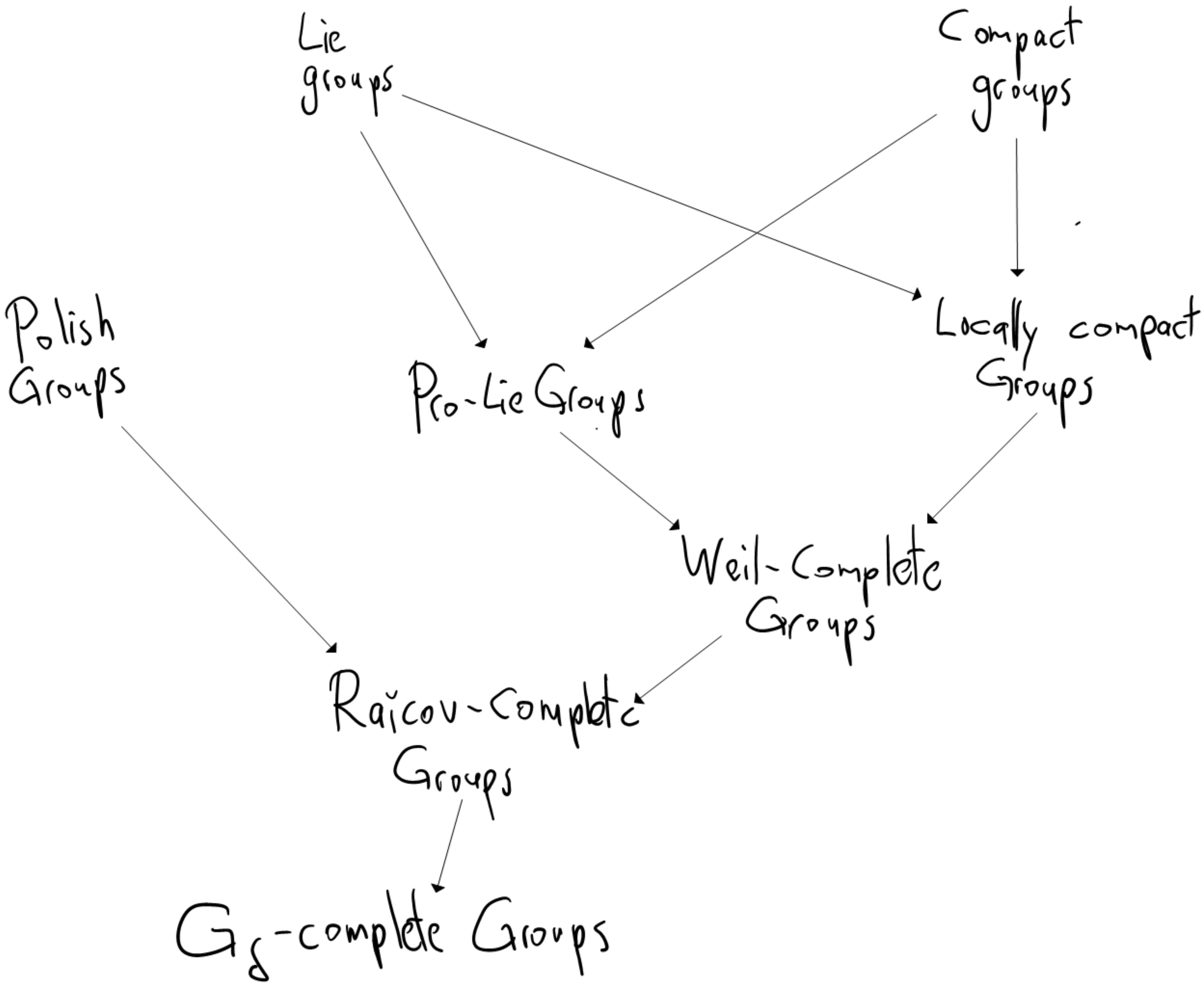
$X = (\mathbb{R}, |\cdot|)$

$\mathrm{Iso}(X) \cong (\mathbb{R}, +) \rtimes (\{-1,1\}, \cdot)$

$X = (\mathbb{R}^n, \|\cdot\|)$

$\mathrm{Iso}(X) \cong (\mathbb{R}^n, +) \rtimes O_n(\mathbb{R})$

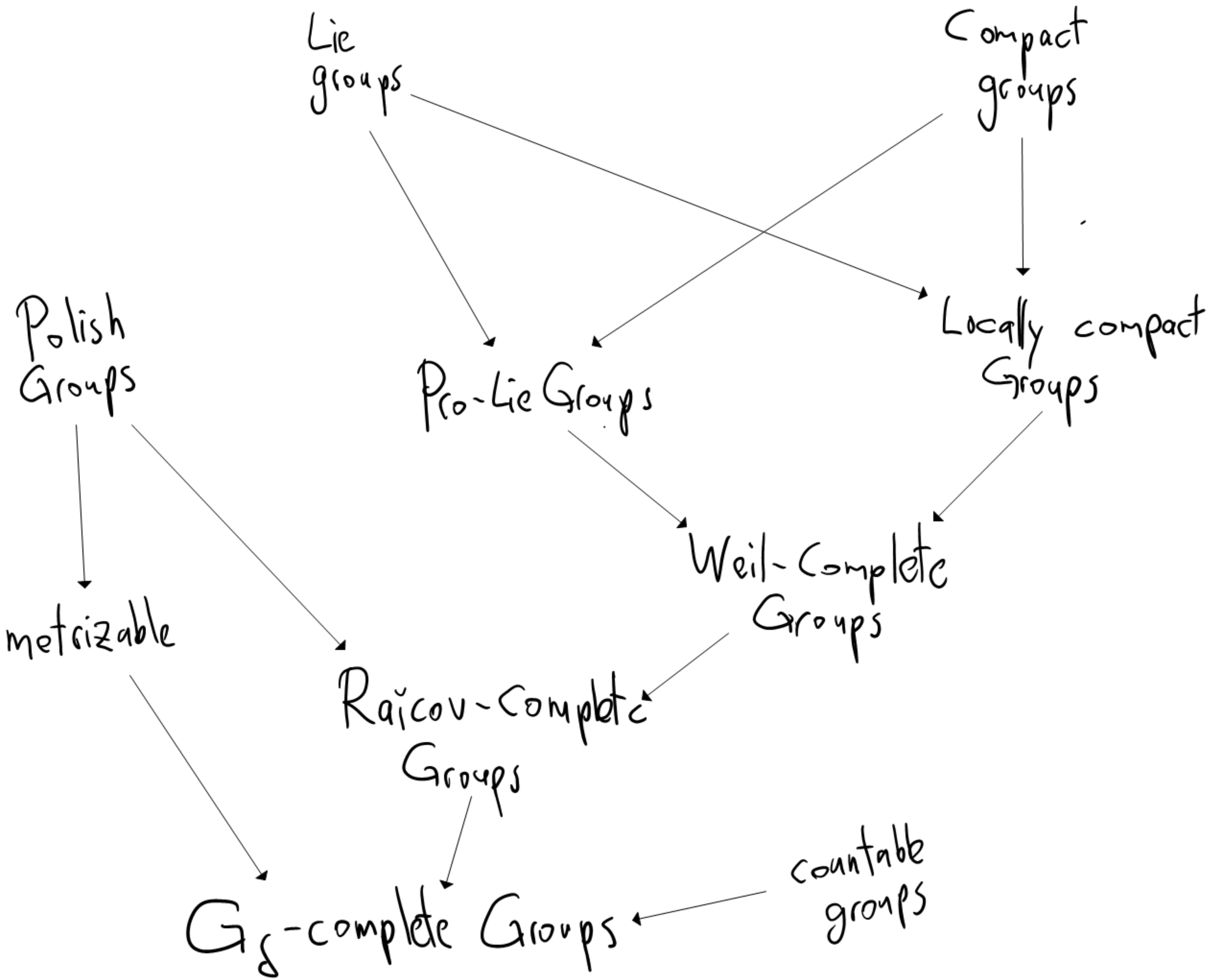
Exercise:
Find (X, d) with $\mathrm{Iso}(X,d) \cong (\mathbb{R}, +)$


Lie groups
Compact groups
Polish Groups
Pro-Lie Groups
Locally compact Groups
Weil-Complete Groups
Raicov-Complete Groups
G_δ-complete Groups

The G_δ-topology

For a top. space $(X, \mathcal{O})$

- a G_δ-set in X is a countable intersection of open sets
- a G_δ-open set in is a union of G_δ-sets
- $\mathcal{O}_{G_\delta} := \{ V \subseteq X \mid V \text{ is } G_\delta\text{-open} \}$ is the G_δ-topology on X
- $(X, \mathcal{O}) \longrightarrow (X, \mathcal{O}_{G_\delta})$ is continuous
- $(G, \mathcal{O})$ topological group $\Longrightarrow$ $(G, \mathcal{O}_{G_\delta})$ topological group
- $(G, \mathcal{O})$ G_δ-complete $:\Longleftrightarrow$ $(G, \mathcal{O}_{G_\delta})$ Raĭkov-complete


Lie groups
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metrizable
Raicov-Complete Groups
countable groups
G_δ-complete Groups

Theorem (Niemiec 2013)

Let G be a topological group.

Then the following are equivalent:

- $\boxed{\left(\exists (X,d) \text{ metric space}\right) \; G \cong \mathrm{Iso}(X,d)}$
- $\boxed{G \text{ is } G_\delta\text{-complete}}$

Counter-Example (Niemiec 2013)

Let κ be an uncountable cardinal.

$$G := \sum_{\alpha<\kappa} \mathbb{R} := \left\{ (x_\alpha) \in \prod_{\alpha<\kappa} \mathbb{R} \;\middle|\; \text{only countable many } x_\alpha \text{ are } \neq 0 \right\}$$

is the Σ-product of κ many real lines.

Then $(G,+)$ is NOT G_δ-complete.

So, this topological group is NOT the isometry group of a metric space!

Summary up to this point:

If you consider

all metric spaces (X,d)

you get "almost all" topological groups $\mathrm{Iso}(X,d)$.

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Now: Can you make sure that

you get "nice" topological groups

by restricting

to "nice" metric spaces ?

Theorem (Myers-Steenrod 1939)

The isometry group of a (connected) Riemannian manifold is a Lie group.

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Fact

(X,d) <u>proper</u> $\Longrightarrow$ $\mathrm{Iso}(X,d)$ locally compact

Theorem (Fukaya-Yamaguchi 1993)

Let $\kappa \in \mathbb{R}$.

The isometry group of a

finite dimensional

geodesic

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with curvature $\geq \kappa$

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(the actual statement is more general)

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(X,d) geodesic

$:\Leftrightarrow \quad \forall a,b \in X, \ d(a,b) = r > 0$

$\Rightarrow \exists \gamma : [0,r] \to X$ isometric embedding with

$\gamma(0) = a$

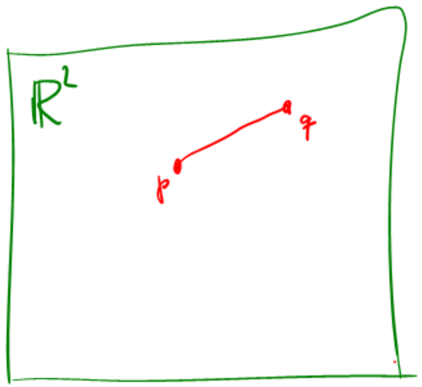
$\gamma(r) = b$.

(X,d) complete

$:\Leftrightarrow$ Every Cauchy sequence converges

Curvature for metric spaces

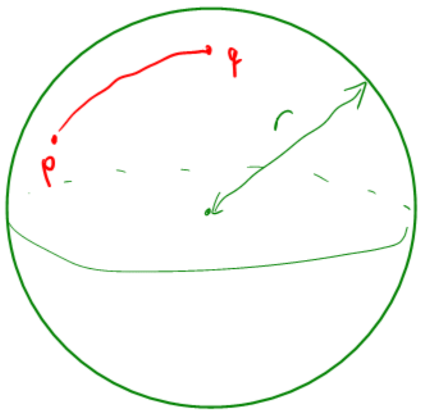
Examples:



Euclidean Plane

curvature $= 0$

"flat"

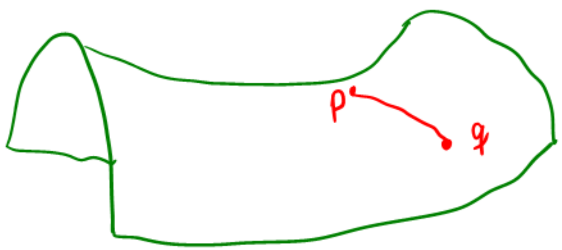


Sphere with Radius $r > 0$

with arclength-metric

Curvature $= \frac{1}{r^2} > 0$

"positive curvature"



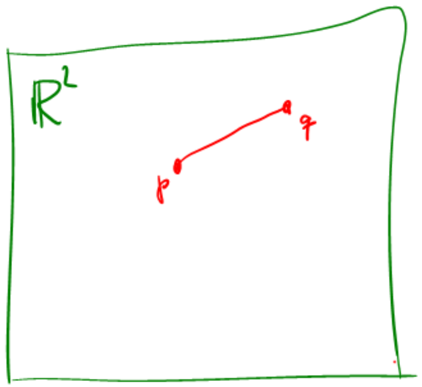
Hyperbolic Plane

Curvature $= -1 < 0$

"negative curvature

Curvature for metric spaces

Examples:



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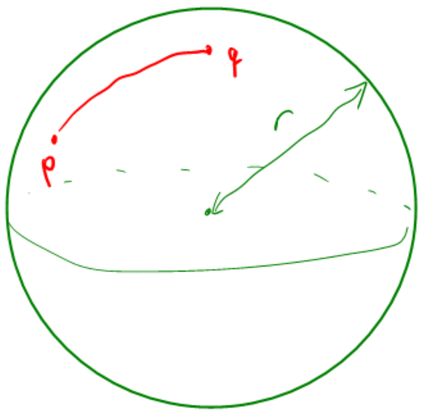

Sphere with Radius $r > 0$

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$$\text{Curvature} = \frac{1}{r^2} > 0$$

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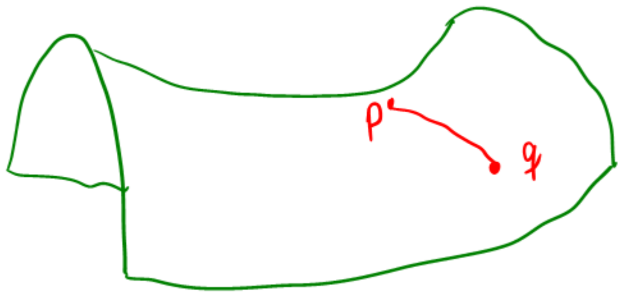


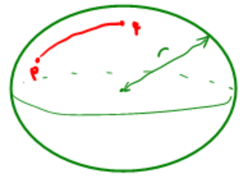
Hyperbolic Plane, scaled by $r > 0$

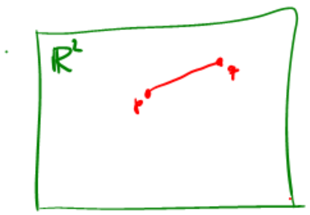
$$\text{Curvature} = -\frac{1}{r^2} < 0$$

"negative curvature

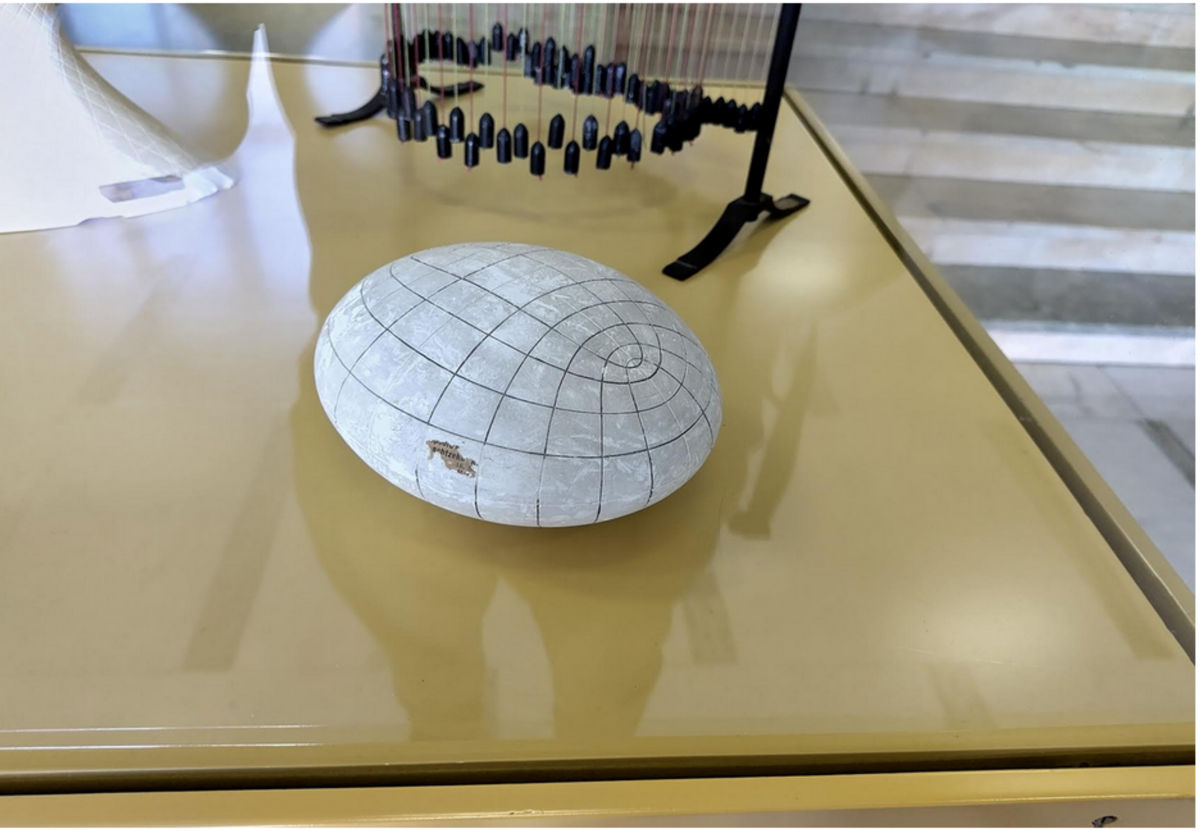
For each $K \in \mathbb{R}$ we set

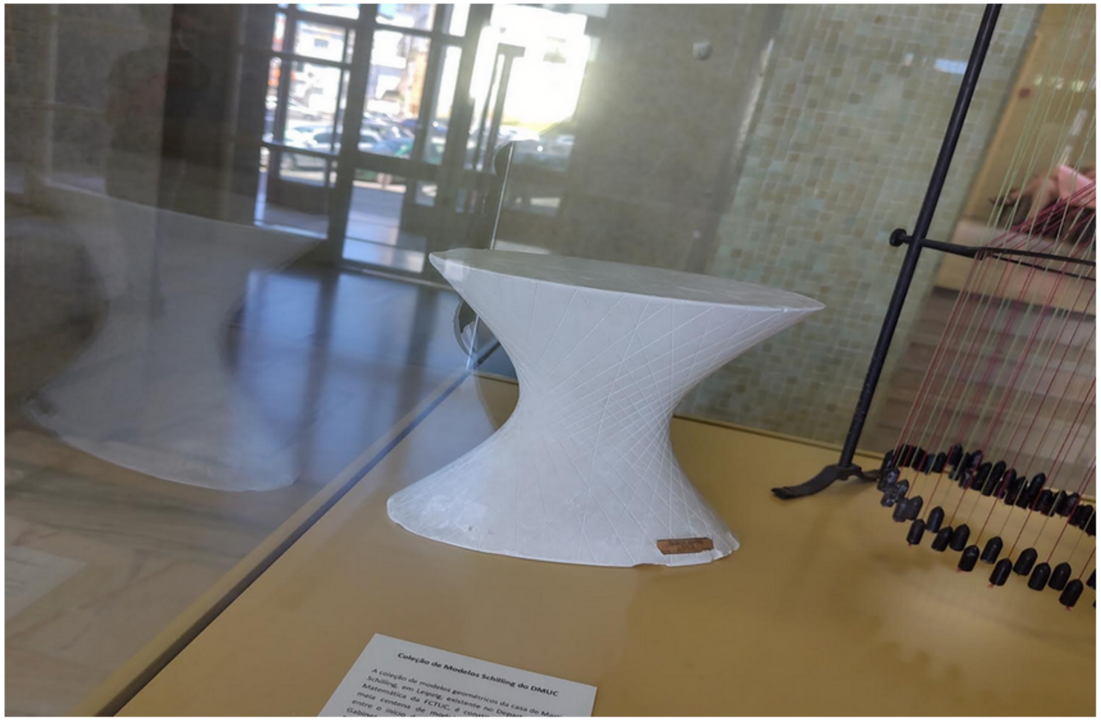
$$M_K := \begin{cases} \text{Sphere with radius } \dfrac{1}{\sqrt{K}} & \text{if } K > 0 \\ \text{Euclidean Plane} & \text{if } K = 0 \\ \text{Hyperbolic Plane, scaled by } \dfrac{1}{\sqrt{-K}} & \text{if } K < 0 \end{cases}$$





"Def": A metric space (X,d) has curvature $\geq K$, if triangles in (X,d) are at least as big as the corresponding reference triangles in M_K.




Coleção de Modelos Schilling do DMUC
A coleção de modelos geométricos da casa de
Schilling, em Leipzig, existente no
Matemática da FCTUC, é
meia centena de
entre o início

Theorem (Fukaya-Yamaguchi 1993)

Let $\kappa \in \mathbb{R}$.

The isometry group of a finite dimensional

Alexandrov space $CBB(\kappa)$ $\left\{ \begin{array}{l} \text{geodesic} \\ \text{complete} \\ \text{metric space} \\ \text{with curvature} \geq \kappa \end{array} \right.$

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"Curvature $\geq \kappa$"

is defined using the reference spaces M_κ.

"finite dimensional"

Hausdorff dimension/top. dim.

What about infinite dimensions?

(X, d) proper Alexandrov space $CBB(\kappa)$ $\left\{ \begin{array}{l} \text{geodesic} \\ \text{complete} \\ \text{metric space} \\ \text{with curvature} \geq \kappa \end{array} \right.$

Is $\mathrm{Iso}(X, d)$ a Lie group?

What about infinite dimensions?

Ex: $X = \prod_{n=1}^{\infty} [-\frac{1}{n}, \frac{1}{n}] \subseteq \ell^2(\mathbb{N})$ "Hilbert cuboid"

$\mathrm{Iso}(X) \cong (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is not a Lie group

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Ex: $X = \prod_{n=1}^{\infty} \frac{1}{n} S^1$ "∞-dim. torus"

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Ex: $X = \ldots$

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group of p-adic integers

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A top. group G is a pro-Lie group iff

$$G \overset{\text{closed}}{\hookrightarrow} \prod_{j \in J} G_j \qquad G_j \text{ Lie groups}$$

$\iff$

$$G = \varprojlim_{\alpha} G_\alpha \qquad \text{for a projective system of Lie groups}$$

$\iff$

G complete and
$\forall U$ 1-nbh of G
$\exists N \subseteq U$ N closed normal subgroup
G/N Lie group

There exists a huge monograph "The Structure of Pro-Lie groups"
(Hofmann / Morris)

Conjecture (D, Nepechiy)

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The isometry group of a

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Alexandrov space CBB(κ)

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Thank you!