Functional Countability vs Exponential Separability

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Outline of the talk

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Definitions and preliminaries

Generalized ordered spaces

Subspaces of products

Separable plus ES

Functionally countable

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(•) Every Lindelöf P space is FC.

(•) Every Lindelöf scattered space is FC.

(•) Any σ-product of copies of $2 = \{0, 1\}$ is FC.

Exponentially separable

Definition (Tkachuk, 2018)

Let X be a space, $\mathcal{F} \subset \wp(X)$ and $A \subset X$. We say that A is **strongly dense** *in $\mathcal{F}$ if $A \cap (\bigcap \mathcal{G}) \neq \emptyset$ for every $\mathcal{G} \subset \mathcal{F}$ with $\bigcap \mathcal{G} \neq \emptyset$.*

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$$\text{ES} \Rightarrow \text{FC}$$

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Proposition (Tkachuk, 2018)

There is a space that is FC but not ES.

Proof.

There exists a MAD family $\mathcal{A} \subset \wp(\omega)$ such that for the ψ-space $\psi(\mathcal{A}) = \omega \cup \mathcal{A}$ it follows that $|\beta\psi(\mathcal{A}) \setminus \psi(\mathcal{A})| = 1$.

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ES vs FC

Question (Tkachuk, 2018)

Let X be a space such that every closed subspace of X is FC. Does it follow that X is ES?

Outline of the talk

GO spaces

Recall that a space X is a **LOTS** if there is a linear order $<$ on X such that the collection of sets of the form

$$\begin{aligned}(\leftarrow, a) &= \{x \in X : x < a\}, \text{ and} \\ (a, \rightarrow) &= \{x \in X : a < x\},\end{aligned}$$

for $a \in X$, is a base for the topology of X. A **GO space** is a subspace of a LOTS.

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Theorem (Tkachuk and Wilson, 2022)

If X is a GO space that is either scattered or locally countable, then the following are equivalent:

(a) X is FC, and

(b) X has countable extent.

GO spaces

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Souslin lines

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Proposition (HG and Spadaro, 2024)

A Souslin line is ES if and only if it is FC.

In a 2023 BIRS-CMO workshop, Alan Dow conjectured that:

Proposition (HG and Spadaro, 2024)

A Souslin line defined from a normal Souslin tree is FC, thus, ES.

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An example with arbitrarily large extent

Theorem (Uspenski, 1984)

If S is an infinite set, there is a dense $X_S \subset [0,1]^S$ that is the union of a countable collection of closed discrete subspaces of size $|S|$.

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Theorem (HG and Spadaro, 2024)

For every uncountable cardinal κ, Uspenski's example of cardinality κ is a FC space with points G_δ, a G_δ diagonal but it has extent κ, so it is not ES.

Function spaces

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Theorem (Tkachuk, 2024)

If X is 0-dimensional and countably compact, then $C_p(X, 2)$ is FC iff X is ω-monolithic (that is, the closure of every countable set is second-countable).

Results about function spaces

For X compact and 0-dimensional we have the following.

$$
\begin{tikzcd}
\substack{X \text{ metrizable} \\ \text{compactum}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "known"] & \substack{X \text{ Eberlein} \\ \text{compactum}} \arrow[r, Rightarrow] \arrow[d, Leftrightarrow, "known*"] & \substack{X \text{ Gul'ko} \\ \text{compactum}} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "{[HG-S,24]}"] & \substack{X\ \omega\text{-monolithic} \\ \text{compactum}} \arrow[d, Leftrightarrow, "{[Tk,24]}"] \\
\substack{C_p(X,2) \\ \text{countable}} \arrow[r, Rightarrow] & \substack{C_p(X,2) \\ \sigma\text{-compact} \\ \text{scattered}} \arrow[r, Rightarrow] & \substack{C_p(X,2) \\ \text{ES}} \arrow[r, Rightarrow] & \substack{C_p(X,2) \\ \text{FC}}
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Question

If X is a 0-dimensional compactum and $C_p(X,2)$ is ES, does it follow that X is a Gul'ko compactum?

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Uncountable, separable and ES

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Let $\mathcal{A} = \{a_\alpha \colon \alpha < \omega_1\} \subset [\omega]^\omega$ be an almost-decreasing family.

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Let $\mathcal{A} = \{a_\alpha : \alpha < \omega_1\} \subset [\omega]^\omega$ be an almost-decreasing family. Let $X = \omega \cup \mathcal{A}$ such that $\omega \cup \{a_0\}$ is open and discrete, and a basic open neighborhood of a_β is of the form

$$\{a_\gamma : \alpha < \gamma \leq \beta\} \cup (a_\alpha \setminus (a_\beta \cup F))$$

for $\alpha < \beta < \omega_1$ and $F \in [\omega]^{<\omega}$. $\square$

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Notice that $\mathcal{A}$ contains a closed but not G_δ subset so the example is not perfectly normal.

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Question

Is there in ZFC an example of an uncountable, separable, ES space that is either perfectly normal or hereditarily separable (or both)?

Question

Is it possible to characterize when an Isbell-Mrowka ψ-space if FC in terms of combinatorial properties of the underlying AD family?

References

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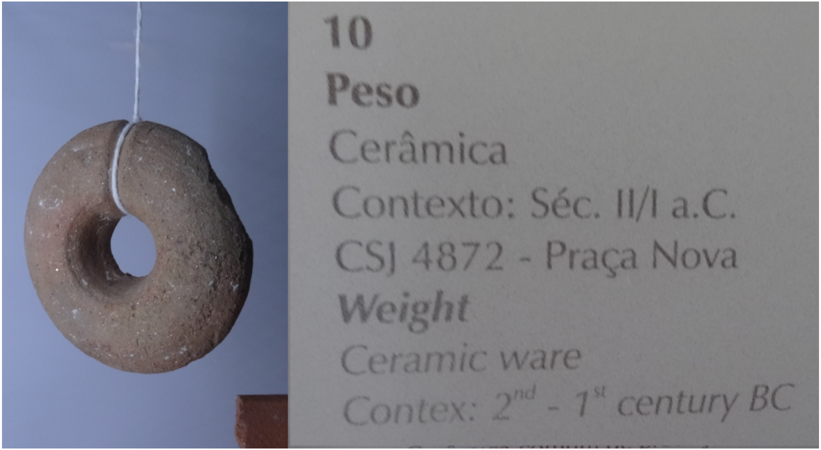
Thank you



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