Equivariant Extension Operators

Ricardo Ramírez Luna

Facultad de Ciencias, UNAM

38th Summer Conference on Topology and its Applications
July 8-12, 2024
Coimbra, Portugal

This is a joint work with Prof. Sergey Antonyan.

Let us begin by establishing the definition of a locally convex linear space.

Definition

A **linear space** V is a real vector space equipped with a Hausdorff topology such that addition and multiplication are continuous functions. In the case that 0 has a basis of convex open neighbourhoods we will say V is a **locally convex linear space**.

A topological space Y is called an absolute extensor for a given space X (notacion $Y \in AE(X)$), if for any closed subset A of X and any continuous function $f : A \to Y$ there exists a continuous extension $F : X \to Y$. In the case that Y is an absolute extensor for any metrizable we will say that Y is an absolute extensor for the class of metrizable spaces.

In 1951 J. Dugundji proved the following result

Theorem (Dugundji)

Let A be a closed subset of the metrizable space Z and W any convex subset of the locally convex linear space V. Then every continuous function $f : A \to W$ admits a continuous extension $F : Z \to W$ such that $Im(F) \subset Conv(Im(f))$.

Now, I will introduce the notation that we will be using throughout this talk.

Let Z be a metrizable space, A a closed subset of Z and V a locally convex linear space. Let $C(Z, V)$ denote the linear space of continuous functions from Z into V and similary for $C(A, V)$. We will equip these vector spaces with the compact-open topology.

If we assign to each continuous function $f \in C(A, V)$ its continuous extension F built by Dugundji, we have a function

$$\Lambda : C(A, V) \to C(Z, V).$$

In 1953, E. Michael observed that Λ is a linear homeomorphic embedding.

We have de following result

Theorem (Simultenous Extension Operators)

Let Z be a metrizable space, A a closed subset of Z and V a locally convex linear space. Then there exists a linear operator

$$\Lambda : C(A, V) \to C(Z, V)$$

such that for every $f \in C(A, V)$ we have the following

1. $\Lambda(f)$ *is an extension of* f
2. $Im(\Lambda(f)) \subset Conv(Im(f))$
3. Λ *is a linear homeomorphic embedding*

This talk is devoted to the equivariant counterpart of this result.

Let G be a topological group.
By a G-space we mean a space X together with a fixed action of the group on X.
Recall that an action is a continuous function $G \times X \to X$ denoted by $(g, x) \to gx$, such that

1. $ex = x$ for each $x \in X$, where e denotes the unity element of G,
2. $g(hx) = (gh)x$ for each $x \in X$ and $h, g \in G$.

Let X be a G-space. For the sets $H \subset G$ and $A \subset X$, we define

$$H(A) = \{ha | h \in H \text{ and } a \in A\}.$$

In the case that $G(A) = A$, we say that A is **invariant**.
For every $x \in X$ we define $G_x = \{g \in G | gx = x\}$, **the stabilizer** of x. In the case that $G_x = G$, we say that x is a G-**fixed point** of X.
A **linear** G-**space** V is a linear space equipped with a continuous action of G, where the action is a linear function.

Let us suppose that X and Y are two G-spaces. We will say that a continuous function $f : X \to Y$ is **equivariant** if $gf(x) = f(gx)$ for every $x \in X$ and $g \in G$.

Remark

If $f : X \to Y$ is an equivariant function between the G-spaces X and Y, and $x \in X$ is a G-fixed point, then $f(x)$ must be a G-fixed point of Y. This follows because, given $gf(x) = f(gx) = f(x)$ for any $g \in G$, we have that $f(x)$ is invariant under the action of G.

This remark implies that no equivariant functions can exist between two G-spaces if the domain X has any G-fixed point and the codomain Y doesn't have any G-fixed point.

Let $G = S^1$ be the circle group and $C(G, \mathbb{C})$ the real Banach space of all continuous functions $f : G \to \mathbb{C}$.
Consider the action of G on $C(G, \mathbb{C})$ defined by

$$(gf)(z) = f(zg).$$

The G-fixed points set of $C(G, \mathbb{C})$ consists precisely of all constant functions.
Now, let $f(z) = e^z$, $z \in G$, and $W = Conv(G(f))$.
Then, the set W doesn't have any G-fixed point.

Let X be a metrizable G-space and let $X^\star$ be the discrete union of X and a singleton $\{*\}$.
The group G acts on $X^\star$ by extending the given action on X and defining $g* = *$ for all $g \in G$.
It is easy to see that $X^\star$ is a metrizable G-space and that X is a closed subset of $X^\star$.
Now, let us consider the identity function $i : X \to X$.
We claim that this function cannot be extended to a function $i' : X^\star \to X$. The reason is that $X^\star$ has a G-fixed point $*$, but X doesn't have any G-fixed point, making impossible to extend i preserving the equivariance.

A topological G-space Y is called an absolute extensor for a given G-space X (notacion $Y \in AE(X)$), if for any closed invariant subset A of X and any equivariant function $f : A \to Y$, there exists an equivariant extension $F : X \to Y$. In the case that Y is an absolute extensor for any metrizable G-space we will say that Y is an absolute extensor for the class of metrizable G-spaces.

The above shows that Dugundji's result is not valid in the class of G-spaces. Therefore, we have been working on analogous results to the theorems of Dugundji and Michael.
To conclude, I will present these results.

Theorem

Let G be a locally compact group, X be a G-space, and V a locally convex linear G-space. Let $C(X,V)$ denote the vector space of continuous functions from X into V endowed with the compact-open topology. Then the action $G \times C(X,V) \to C(X,V)$ defined by

$$(g,f) \to gf; \qquad (gf)(x) = gf(g^{-1}x); \forall x \in X$$

is a continuous linear action.

Theorem

Let G be a compact Lie group, Z a metrizable G-space, A a closed invariant subset of Z, and V a locally convex linear G-space. There exist an invariant neighborhood X of A in Z and an equivariant linear operator

$$\Lambda : C(A, V) \to C(Z, V)$$

such that for every $f \in C(A, V)$, we have the following

1. $\Lambda(f)$ *is an extension of* f,
2. $Im(\Lambda(f)) \subset Conv(Im(f) \cup \{0\})$ *and* $Im(\Lambda(f)|_X) \subset Conv(Im(f))$,
3. Λ *is a linear equivariant homeomorphic embedding provided* $C(A, V)$ *and* $C(Z, V)$ *both carry the linear action defined above.*

Remark

Let $f \in C(A, V)$ be an equivariant function. Then for $x \in A$ and $g \in G$, we have

$$f(gx) = gf(x) \iff g^{-1}f(gx) = f(x) \iff (g^{-1}f)(x) = f(x) \iff g^{-1}f = f.$$

Then f is equivariant iff f is a G-fixed point of $C(A, V)$ (under the action given in the previous theorem).

Theorem

Let G be a compact Lie group, Z a metrizable G-space and A a closed invariant subset of Z. For a locally convex linear G-space V denote by $E(Z,V)$ the vector space of all equivariant fuctions $f : Z \to V$ endowed with the compact-open topology. Then there exist an invariant neighborhood X of A in Z and a (non-equivariant) linear operator

$$L : E(A,V) \to E(Z,V)$$

such that for every $f \in E(A,V)$, we have the following

1. *$L(f)$ is an extension of f,*
2. *$Im(L(f)) \subset Conv(Imf \cup \{0\})$ and $Im(L(f)|_X) \subset Conv(Imf)$,*
3. *L is a linear homeomorphic embedding.*

Thank you!