Effective descent morphisms: reflection and preservation

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Outline

- Effective descent morphisms.
- Example 1: Lax comma categories.
- Example 2: Artin gluing.
- Preservation results.

Effective descent morphisms

Examples

- If $C = \mathsf{Top}, \mathsf{Loc}, \mathsf{open surjections are e.d.m.}$
- If $C = \mathsf{Top}, \mathsf{Loc}, \mathsf{proper surjections are e.d.m.}$

Regular epimorphisms

Lemma (Janelidze, Tholen 1994)

Regular epimorphisms in ${\mathcal A}$ (quotient maps) are effective descent morphisms, when

- \mathcal{A} is a Barr-exact category;
- \mathcal{A} is a locally cartesian closed category.

Reflection result

Theorem (Janelidze, Tholen 1994)

Let $F \colon \mathcal{C} \to \mathcal{D}$ be fully faithful and pullback preserving.

Let $p \colon x \to y$ such that Up is an effective descent morphism.

Then p is an e.d.m. if and only if, for all pullbacks of the form

$$\begin{array}{ccc} Uw & \longrightarrow d \\ \downarrow & & \downarrow \\ Ux & \longrightarrow Uy \end{array}$$

we have $d \cong Uz$.

Examples

Lemma (Reiterman, Tholen, 1994)

E.d.m.s of topological spaces were characterized via the embedding

 $\mathsf{Top} \to \mathsf{PsTop}$

Lemma (Janelidze, 1990s)

E.d.m.s of categories were characterized via the embedding

 $\mathsf{Cat} \to [\Delta^\mathsf{op}_3,\mathsf{Set}]$

Lemma (P. 2024) E.d.m.s in $Fam(\mathcal{C})$ were studied via

 $\mathsf{Fam}(\mathcal{C}) \to \mathsf{Set}$

Families of objects

Let \mathcal{A} be a category. Fam(\mathcal{C}) consists of:

Objects: set-indexed families $(a_j)_{j \in J}$ of \mathcal{A} -objects. Morphisms $(a_j)_{j \in J} \to (b_k)_{k \in K}$: a function $f: J \to K$, a family $\phi = (\phi_j: a_j \to b_{f(j)})_{j \in J}$.

We will consider the following:

$$Ord//X$$
, $Cat//A$

for X an ordered set, \mathcal{A} a category.

Lemma (C., L.N. 2023)

The forgetful functor $\operatorname{Ord}//X \to \operatorname{Ord}$ preserves effective descent morphisms, provided X has a bottom element and downsets have binary meets.

Lemma (C., L.N., P., 2024)

The forgetful functor $Cat//A \rightarrow Cat$ preserves effective descent morphisms, provided A has a initial object (either strict, or a zero object) and pullbacks.

Lemma (C., L.N., 2023; C. P. 2024)

If the down-sets of X are complete, then the forgetful functor $\operatorname{Ord}//X \to \operatorname{Set}//X$ maps pullback-stable regular morphisms to effective descent morphisms.

Artin gluing

Artin gluing

Artin gluing Gl(F) of a functor $F: \mathcal{A} \to \mathcal{B}$: Objects: morphisms $f: b \to Fa$, Morphisms $f \to f'$: pairs $\beta: b \to b'$, $\alpha: a \to a'$ such that



commutes.

Preservation result

Let $F: \mathcal{A} \to \mathcal{B}$ be a pullback-preserving functor between categories with pullbacks.

Lemma (C., L.N., R.)

If ${\mathcal B}$ has a strict initial object, then

$$\operatorname{Gl}(F) \to \mathcal{A}$$

preserves effective descent morphisms.

Preservation results

General result

Lemma (C., L.N., P., 2024)

If we have an adjunction



with L fully faithful, then under suitable conditions,

- the induced comonad LU preserves effective descent morphisms,
- L reflects effective descent morphisms,

thereby concluding that ${\cal U}$ preserves effective descent morphisms.

Thank you for your time!