Endpoint-homogeneous smooth fans

Rene Gril Rogina

Joint work with Will Brian, UNC Charlotte

University of Maribor, Slovenia

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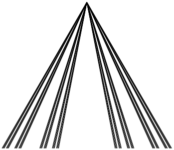
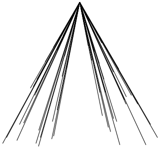
$\frac{1}{n}$-homogeneous fans

Definition

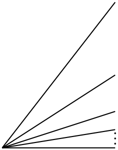
Let F be a fan and n be a positive integer. We say that F is $\frac{1}{n}$-*homogeneous* if the action of its homeomorphism group has exactly n distinct orbits.



Cantor fan



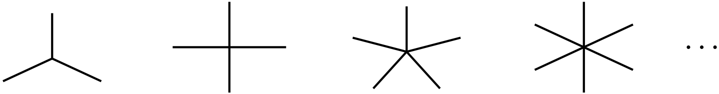
Lelek fan



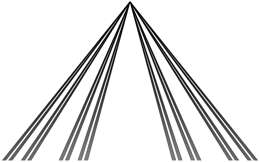
Harmonic fan

Acosta, Hoehn, Juárez give a full classification of $\frac{1}{3}$-homogeneous smooth fans. They also prove that there is no $\frac{1}{4}$-homogeneous smooth fan.

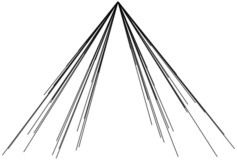
simple n-ods



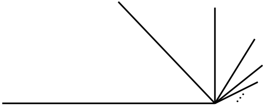
Cantor fan



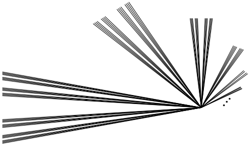
Lelek fan



star fan



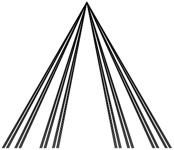
the star of Cantor fans

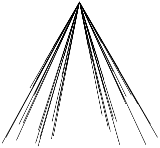


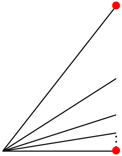
Endpoint-homogeneous fans

Definition

Let F be a fan and $E(F)$ denote the set of its endpoints. We say that F is *endpoint-homogeneous* if for any $e, e' \in E(F)$ there exists a homeomorphism $h : F \to F$ such that $h(e) = e'$.


Cantor fan


Lelek fan


Harmonic fan

Endpoint-generated fans

Definition

Let F be a fan with top v.

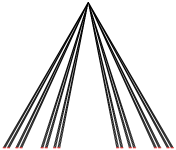
1. For any two distinct points $x, y \in F$ we use $B[x, y]$ to denote the unique arc in F with endpoints x and y.
2. For any endpoint $e \in E(F)$, we call $B[v, e]$ a *blade of the fan F*.
3. We use $\mathcal{B}(F) = \{B[v, e] \mid e \in E(F)\}$ to denote the set of all blades of the fan F.

Definition

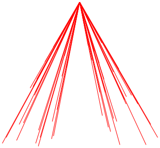
Let F be a fan and $X \subseteq [0, 1]$. We say that F is X-endpoint-generated (X-EPG) if for any blade $B = B[v, e] \in \mathcal{B}(F)$ there exists a homeomorphism $\varphi : [0, 1] \to B[v, e]$ such that

1. $\varphi(0) = v$ and
2. $\varphi(X) = B \cap \overline{(E(F) \setminus B)}$

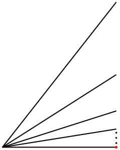
Endpoint-generated fans



Cantor fan



Lelek fan



Harmonic fan

Problem

For which $X \subseteq [0, 1]$ does there exist an X-EPG smooth fan? What can be said about the relationship between $\frac{1}{n}$-homogeneity, endpoint-homogeneity and endpoint-generation?

Endpoint-generated

Endpoint-homogeneous

$\frac{1}{n}$-homogeneous

Problem

For which $X \subseteq [0,1]$ does there exist an X-EPG smooth fan? What can be said about the relationship between $\frac{1}{n}$-homogeneity, endpoint-homogeneity and endpoint-generation?

$$\begin{tikzcd}
 & \text{Endpoint-generated} & \\
\text{Endpoint-homogeneous} \arrow[ur, Rightarrow] & & \frac{1}{n}\text{-homogeneous}
\end{tikzcd}$$

Theorem

Let $X \subseteq [0,1]$, $X \neq \emptyset$, and suppose F is an X-EPG smooth fan.

1. *If $1 \notin X$, then $E(F)$ is countable.*
2. *If $0 \notin X$, then $E(F)$ is uncountable.*

Theorem

If $X \subseteq [0,1]$, and there is an X-EPG smooth fan, then either $X = \emptyset$, or else $0 \in X$ or $1 \in X$ (or both).

Note that there are examples of X-EPG fans for sets X satisfying each of the four possibilities:

1. $X = \emptyset$: The simple n-od, for any $n \geq 3$, is $\emptyset$-EPG.
2. $0, 1 \in X$: The Lelek fan is $[0,1]$-EPG.
3. $1 \in X$: The Cantor fan is $\{1\}$-EPG.
4. $0 \in X$: The star is $\{0\}$-EPG.

The classification of X-EPG smooth fans

Suppose $X \subseteq [0,1]$ and there is an X-EPG fan. Then one of the following must hold:

1. $X = \emptyset$, $\{1\}$, or $[0,1]$, or
2. X is a closed subset of $[0,1]$ with $0 \in X$ and $1 \notin X$, or
3. X is a closed subset of $[0,1]$ with $0, 1 \in X$, and 1 is an isolated point of X (previous case $\cup\{1\}$).

Theorem

Suppose $X \subseteq [0,1]$ and there is an X-EPG smooth fan. If $0 \notin X$ and $1 \in X$, then $X = \{1\}$.

Theorem

Suppose $X \subseteq [0,1]$ with $1 \in X$, and there is an X-EPG smooth fan. Then either 1 is an isolated point of X, or $X = [0,1]$.

Fences and combs of fans

Let $A \subseteq [0,1]^2$ and $y \in [0,1]$. We use A_y to denote the set $A_y = \pi_1(A \cap ([0,1] \times \{y\})) \subseteq [0,1]$.

Definition

Let F be a smooth fan with top v and $A \subseteq [0,1]^2$.

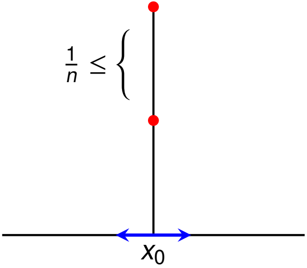
1. We say that A is a *fence* if
   1. $A_0 = \emptyset$.
   2. For each $y \in (0,1]$, A_y is totally disconnected.
   3. For each $y_1, y_2 \in (0,1]$, if $y_1 \le y_2$, then $A_{y_2} \subseteq A_{y_1}$.
2. We say that A is a *fence of the fan F* if A is homeomorphic to $F \setminus \{v\}$.

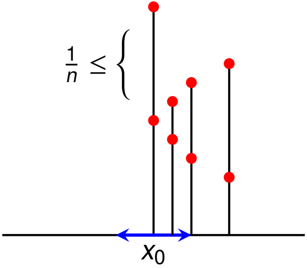
Definition

Let F be a fan and let A be a fence of the fan F. We call $A \cup ([0,1] \times \{0\})$ a *comb of the fan F*.

Theorem

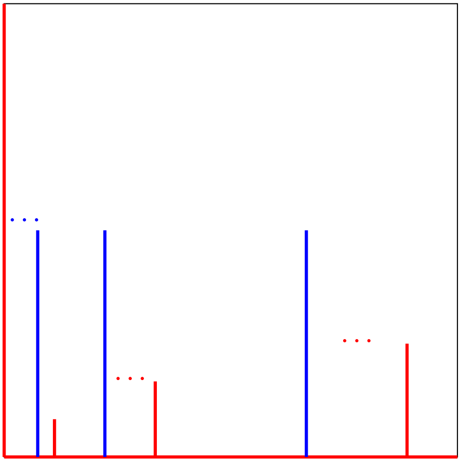
Suppose $X \subseteq [0,1]$ *and there is an* X*-EPG smooth fan. If* $0 \notin X$ *and* $1 \in X$*, then* $X = \{1\}$*.*





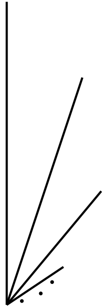
Comb for $X \cap \{0,1\} = \{0\}$

Let $X \subseteq [0,1]$ be s.t. $X \cap \{0,1\} = \{0\}$. Let (y_n) be a sequence in $[0,1]$ s.t. its set of limit points is either X or $X \setminus \{0\}$.



For $X \cap \{0,1\} = \{0,1\}$

Let $X \subseteq [0,1]$ be s.t. 1 is isolated in X. Construct the Y-EPG fan for $Y = X \setminus \{1\}$ using the given construction.



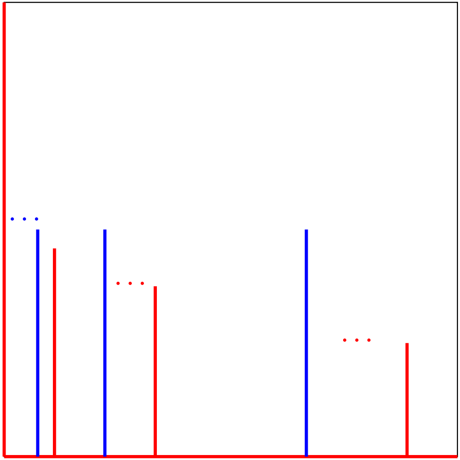
For $X \cap \{0,1\} = \{0,1\}$

Let $X \subseteq [0,1]$ be s.t. 1 is isolated in X. Construct the Y-EPG fan for $Y = X \setminus \{1\}$ using the given construction.

Result

Theorem

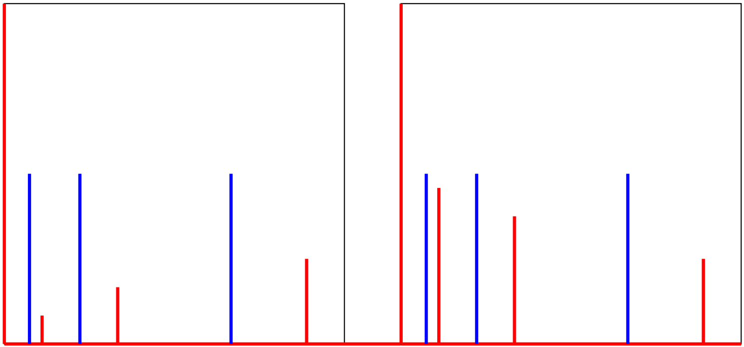
Let $X \subseteq [0,1]$ s.t. there exists an X-EPG smooth fan. Then there exists an endpoint-homogeneous X-EPG smooth fan.



Result

Theorem

Let $X \subseteq [0,1]$ s.t. there exists an X-EPG smooth fan. Then there exists an endpoint-homogeneous X-EPG smooth fan.



Result

Theorem

Let $X \subseteq [0,1]$ *s.t. there exists an* X*-EPG smooth fan. Then there exists an endpoint-homogeneous* X*-EPG smooth fan.*

Theorem

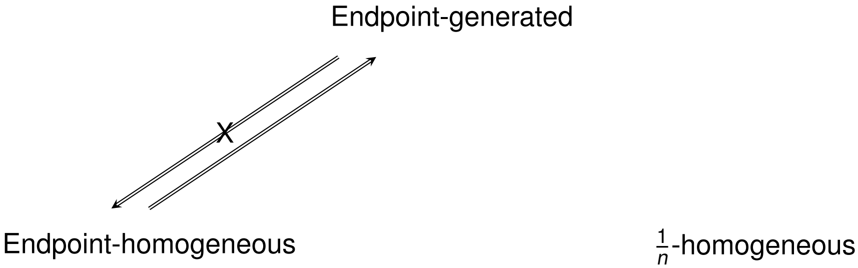
Let $X \subseteq [0,1]$ *s.t. there exists an* X*-EPG smooth fan and* $X \notin \{\emptyset, \{0\}, \{1\}, \{0,1\}, [0,1]\}$*. Then there exists a non-endpoint-homogeneous* X*-EPG smooth fan.*

Theorem

Let $X \in \{\{0\}, \{1\}, \{0,1\}, [0,1]\}$*. The* X*-EPG smooth fan is unique up to homeomorphism.*

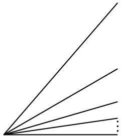
Problem

For which $X \subseteq [0, 1]$ does there exist an X-EPG smooth fan? What can be said about the relationship between $\frac{1}{n}$-homogeneity, endpoint-homogeneity and endpoint-generation?



$\frac{1}{n}$-homogeneity

1. There exist $\frac{1}{n}$-homogeneous smooth fans that are not endpoint-homogeneous (and therefore not EPG):



2. There exist EPG (both endpoint-homogeneous and non-endpoint-homogeneous) smooth fans that are not $\frac{1}{n}$-homogeneous (choose $X = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\} \cup \{0\}$).

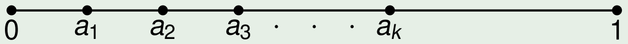
We have examples of $\frac{1}{3}$- and $\frac{1}{5}$-homogeneous smooth fans, we know that $\frac{1}{4}$-homogeneous smooth fans do not exist. Can we give an example for each $n \geq 5$?

For odd $n \geq 5$

If a fan is endpoint-homogeneous, then each orbit from the action of the homeomorphism group must intersect each blade of the fan. With this in mind, it is simple to give examples of X such that the constructed X-EPG smooth fan is $\frac{1}{n}$-homogeneous.

Example

For odd $n = 2k + 1 \geq 5$, we choose some $a_1, a_2, \ldots, a_k$ such that $0 < a_1 < \cdots < a_k < 1$, and let $X = \{0, a_1, \ldots, a_k\}$.



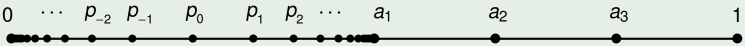
For even $n \geq 5$

Example

Let $\{p_i\}_{i \in \mathbb{Z}}$ be a subset of $[0, 1]$ such that $p_i < p_j$ whenever $i < j$, and $\lim_{i \to -\infty} p_i = 0$, and $\lim_{i \to \infty} p_i < 1$. Let $a_1 = \lim_{i \to \infty} p_i$, and if $n \geq 8$ fix some $a_2 < \cdots < a_m < 1$, where $m = \frac{n-4}{2}$. Let

$$X = \{p_i\}_{i \in \mathbb{Z}} \cup \{0, a_1\} \cup \{a_2, \ldots, a_m\}.$$

This set X is illustrated below for $n = 10$.



Results

Theorem

For every $n \geq 5$ there exists a smooth fan that is $\frac{1}{n}$-homogeneous, endpoint-homogeneous and EPG.

Theorem

For every $n \geq 5$ there exists a smooth fan that is $\frac{1}{n}$-homogeneous, but is not endpoint-homogeneous (and therefore not EPG).

For larger n it becomes difficult to fully classify all $\frac{1}{n}$-homogeneous smooth fans. Can anything be said for $n = 5$ or $n = 6$?

There exist non-smooth fans that are X-EPG for subsets X that do not appear in smooth fans (for example $[\frac{1}{2}, 1]$). Can we obtain a similar classification for non-smooth fans?

Thank you!