Choquet simplex of invariant measures for minimal homeomorphisms on manifolds

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Based on joint work with Jernej Činč, Till Hauser, Dominik Kwietniak and Piotr Oprocha

Dynamical system

Topological dynamical system

- X - a compact metric space, $T\colon X \to X$ is continuous.
- $\mathcal{M}_T(X)$ - the set of all T-invariant probability measures.

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- Every $k \in K$ is a unique generalized convex combination of its extreme points.
- $\mathcal{M}_T(X)$ is a Choquet simplex with ergodic measures as extreme points.

Motivation

What class of dynamical systems can realise every Choquet simplex as a set of its invariant measures?

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Theorem (Downarowicz 1991)

For every Choquet simplex K there exists a **minimal subshift** (X, T) such that $\mathcal{M}_T(X) \approx K$.

Here, $A \approx B$ means A is affinely homeomorphic to B.

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Given a Choquet simplex K, can we realise it as a set of invariant measures for **minimal homeomorphism on a manifold**?

Main theorem

Theorem A (B., Činč, Hauser, Kwietniak, Oprocha)

Let $\mathbb{M}$ be a compact manifold of dimension $d \geq 2$ with a minimal and uniquely ergodic homeomorphism F and K be a Choquet simplex. Then there exists a minimal homeomorphism $f_K : \mathbb{M} \to \mathbb{M}$ such that $\mathcal{M}_{f_K}(\mathbb{M}) \approx K$.

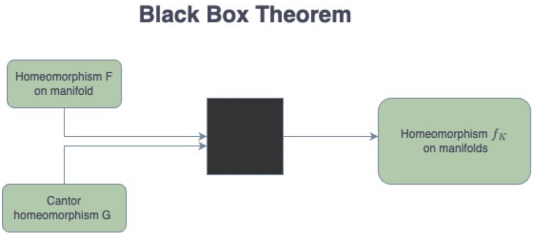
- $\mu \in \mathcal{M}_T(X)$ is **ergodic** if every T-invariant Borel subset $B \subseteq X$ has measure 0 or 1.
- (X, T) is a **minimal system** if X is the only non-empty closed T-invariant subset of X.

Black Box theorem

Theorem (Beguin, Crovisier, Le Roux)

Let F be a minimal and uniquely ergodic homeomorphism on a compact manifold $\mathbb{M}$ of dimension $d \geq 2$ and G be a homeomorphism on some Cantor space C. Then there exists a minimal homeomorphism $\tilde{F} : \mathbb{M} \to \mathbb{M}$ which is universally isomorphic to $F \times G$.

- In particular $\mathcal{M}_{\tilde{F}}(\mathbb{M}) \approx \mathcal{M}_{F \times G}(\mathbb{M} \times C)$



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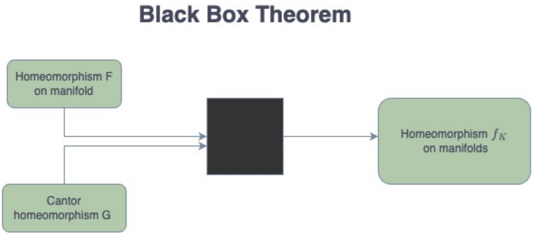

Reduced theorem

Theorem B (B., Činč, Hauser, Kwietniak, Oprocha)

For every Choquet simplex K and $\alpha \notin \mathbb{Q}$, there exists a Cantor homeomorphism G such that

- (C, G) is a topological extension of the irrational rotation R_α,

Definition

(X, T) is a **topological extension** of (Y, S) if there exists $\varphi : X \xrightarrow[\text{continuous}]{\text{onto}} Y$ such that $\varphi \circ T = S \circ \varphi$.

Reduced theorem

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- (C, G) is a topological extension of the irrational rotation R_α,
- $\mathcal{M}_G(C) \approx K$,
- For every $\mu \in \mathcal{M}^e_G(C)$, the measure preserving systems (C, G, μ) and $(\mathbb{T}, R_\alpha, \lambda_{\mathbb{T}})$ are isomorphic.

How Black Box works?

- Choose an irrational $\alpha \notin \mathrm{Spec}(\mathbb{M}, F, \nu)$.
- Using Theorem B construct the topological extension of R_α.
- For every $\mu \in \mathcal{M}_G^e(C)$ the systems (C, G, μ) and $(\mathbb{M}, F, \nu)$ are disjoint.
- Hence $\mathcal{M}_{\tilde{F}}(\mathbb{M}) \approx \mathcal{M}_{F\times G}(\mathbb{M}\times C) \approx \mathcal{M}_F(\mathbb{M}) \times \mathcal{M}_G(C) \approx K$.

Ingredients for the construction

- An irrational rotation R_α on the circle.

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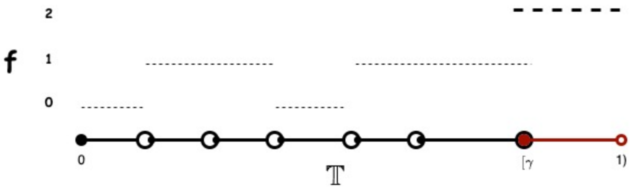
Definition (Marked semicocycles)

A **marked semicocycle** $f : \mathbb{T} \to \{0, 1, 2\}$ is a function such that

- $f^{-1}(\{2\}) = [\gamma, 1)$,
- has at most countably many discontinuities.



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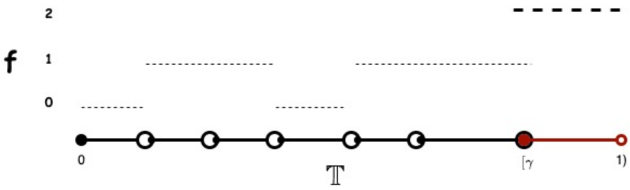

- Minimal and uniquely ergodic subshift X_f over $\{0, 1, 2\}$.

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Every Choquet simplex is affinely homeomorphic to the intersection of a decreasing sequence of Bauer simplices in some locally convex linear space.

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- Hence we get a decreasing sequence K_n such that $K = \cap_{n\in\mathbb{N}} K_n$.
- If K_n is a Bauer simplex then $K_n \approx \mathcal{M}(ext(K_n))$.

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Hence $\forall\ z \in K$

$$\lim_{n\to\infty} \pi_n(z) = \pi_\infty(z)$$

exist and is a continuous embedding from $K \to \mathcal{M}_\sigma(\{0,1,2\}^{\mathbb{Z}})$.

- The limit of this inductive construction π_∞ is an affine embedding into $\mathcal{M}_\sigma(\{0,1,2\}^{\mathbb{Z}}$.
- At every step n we also construct X_n such that $\mathcal{M}_\sigma(X_n) \approx K_n$.
- Then we prove that there exists a subshift $X_\infty \subseteq \{0,1,2\}^{\mathbb{Z}}$ defined by

$$X_\infty = \bigcap_{N=1}^{\infty} \overline{(\bigcup_{n=N}^{\infty} X_n)}$$

  such that $\mathcal{M}_\sigma(X_\infty) \approx K$.
- (X_∞, σ) is a topological extension of R_α.
- $\forall \mu \in \mathcal{M}_\sigma^e(X_\infty)$ the system (X_∞, σ, μ) is isomorphic to $(\mathbb{T}, R_\alpha, \lambda_{\mathbb{T}})$.

Thank you for your attention