Bohr compactification of topological groups and Schur ultrafilters

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SUMTOPO24

Coimbra, 09.07.2024

Definition

Let X be a discrete space. Stone-Čech compactification βX is the set of all ultrafilters on X endowed with the topology generated by the base $\{\langle A\rangle : A \subseteq X\}$, where $\langle A\rangle = \{\mathbf{u} \in \beta X : A \in \mathbf{u}\}$.

Definition

If S is a discrete semigroup, then the semigroup operation on S can be canonically lifted to the semigroup operation on βS as follows: if $\mathbf{u}, \mathbf{v} \in \beta S$, then $\mathbf{uv}$ is a filter generated by the base consisting of the sets $\bigcup_{x\in U} xV_x$, where $U \in \mathbf{u}$ and $\{V_x : x \in U\} \subset \mathbf{v}$ are arbitrary.

We used the word canonically, because the defined above semigroup operation on βS is unique among those extending the operation of S and satisfying the following two natural conditions:

(i) for each $u \in \beta S$ the right shift $\rho_u : \beta S \to \beta S$, $\rho_u(x) = xu$ is continuous;

(ii) for each $s \in S$ the left shift $\lambda_s : \beta S \to \beta S$, $\lambda_s(x) = sx$ is continuous.

A semigroup S endowed with a topology is called right topological if all right shifts in S are continuous. An element $e \in S$ is called idempotent if $ee = e$.

Theorem (Ellis-Numakura)

Each compact Hausdorff right topological semigroup contains an idempotent.

βS contains at least $2^{\mathfrak{c}}$ idempotents for each infinite discrete semigroup S.

Observation

An ultrafilter $\mathbf{u}$ on a semigroup S is an idempotent if and only if for each $U \in \mathbf{u}$ there exists $x \in U$ and $U \supset V_x \in \mathbf{u}$ such that $xV_x \subseteq U$.

The notion of an idempotent ultrafilter can be naturally weakened as follows:

Definition

An ultrafilter $\mathbf{u}$ on a semigroup S is called

- Schur if for any $U \in \mathbf{u}$ there exist $x, y \in U$ such that $xy \in U$;
- infinitary Schur if for any $U \in \mathbf{u}$ there exist $x \in U$ and an infinite subset $A \subset U$ such that $xA \subseteq U$.

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If G is a group, then for each $\mathbf{u} \in \beta G$ let $\mathbf{u}^{-1} = \{U^{-1} : U \in \mathbf{u}\}$.
The following two results provide examples of Schur ultrafilters.

Proposition (Protasov)

For any ultrafilter $\mathbf{u}$ on a group G, $\mathbf{u}\mathbf{u}^{-1}$ is a Schur ultrafilter.

Proposition (B., Zlatoš)

Let G be a group. Then for each $\mathbf{u} \in \beta G$ and idempotent $\mathbf{e} \in \beta G$ the ultrafilters $\mathbf{u}\mathbf{e}\mathbf{u}^{-1}$ and $\mathbf{u}^{-1}\mathbf{e}\mathbf{u}$ are infinitary Schur.

Schur ultrafilters

Proposition (Protasov)

For any Schur ultrafilter $\mathbf{u}$ on a group G, and $U \in \mathbf{u}$ we have $UU^{-1} \in \mathbf{u}$.

Proposition (B., Zlatoš)

For any ultrafilter u on a group G the following conditions are equivalent:

- u is a Schur ultrafilter;
- $UU \in u$ for each $U \in u$;
- $UU^{-1} \in u$ for each $U \in u$.

Proposition (B., Zlatoš)

Let G be a discrete group, S be a Hausdorff topological semigroup and $f : \beta G \to S$ be a continuous homomorphism. Then $f(\mathbf{u})$ is idempotent for every Schur ultrafilter $\mathbf{u}$. Moreover, for each Schur ultrafilter $\mathbf{u} \in \beta G$ there exists an idempotent ultrafilter $\mathbf{e} \in \beta G$ such that $f(\mathbf{u}) = f(\mathbf{e})$.

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Schur ultrafilters

By $\mathrm{Sch}(G)$ ($\mathrm{Sch}^\infty(G)$, resp.) we denote the set of all Schur (infinitary Schur, resp.) ultrafilters on a discrete group G.

Fact 1 (B., Protasov, Zlatoš)

For any group G the sets $\mathrm{Sch}(G)$ and $\mathrm{Sch}^\infty(G)$ are closed nowhere dense in βG.

Fact 2 (B., Protasov, Zlatoš)

For any commutative group G, $\mathrm{Sch}(G)$ and $\mathrm{Sch}^\infty(G)$ are subsemigroups of βG.

Fact 3 (B., Zlatoš)

Let $\mathbf{u}$ be a Schur ultrafilter on $\mathbb{Z}$. Then for each $U \in \mathbf{u}$ and $n \in \omega$ there exists $x \in U$ such that $|\{y \in U : x + y \in U\}| \geq n$.

Fact 4 (B., Zlatoš)

Let $\mathbf{u}$ be an ultrafilter on a group G. Then the following assertions hold:

- if $\mathbf{u}$ is idempotent, then $\mathbf{u}$ is not a weak P-point (Folklore);
- if $\mathbf{u}$ is infinitary Schur, then $\mathbf{u}$ is not a P-point;
- if $\mathbf{u}$ is a Schur ultrafilter on $\mathbb{Z}$, then $\mathbf{u}$ is not selective.

Bohr compactification

A Bohr compactification of a topological group G is a pair $(b, \mathfrak{b}G)$ such that $\mathfrak{b}G$ is a compact topological group, $b\colon G \to \mathfrak{b}G$ is a continuous homomorphism, and if $g\colon G \to T$ is a continuous homomorphism to a compact topological group T, then there exists a unique continuous homomorphism $f\colon \mathfrak{b}G \to T$ such that the diagram commutes:

$$\begin{tikzcd} G \arrow[r, "b"] \arrow[d, "g"'] & \mathfrak{b}G \arrow[dl, "f"] \\ T & \end{tikzcd}$$

Description of Bohr compactification

Despite being a very useful tool in the study of topological groups, the structure of Bohr compactification is far from being clear. Although, it is described for locally compact commutative groups.

Theorem (Folklore)

Let G be a locally compact commutative topological group. Then $\mathfrak{b}G$ is topologically isomorphic to the dual group of the group $\widehat{G}$ that is endowed with the discrete topology.

Recall that the dual group $\widehat{G}$ of a topological group G is the set of all continuous homomorphisms from G into the circle $\mathbb{T}$ endowed with the compact-open topology and pointwise multiplication.

Another approach to description of Bohr compactification was initiated by Zlatoš. Namely he showed the following:

Theorem (Zlatoš)

Let G be a discrete commutative group. Then $\mathfrak{b}G$ is topologically isomorphic to the quotient semigroup $\beta G/\rho$, where ρ is the smallest closed congruence merging all Schur ultrafilters to the unit of G.

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Bohr compactification

If G is a topological group, then by G_d we denote the group G endowed with the discrete topology. For a topological group G let Ψ be the least closed congruence on βG_d such that

$$\{(\mathbf{u}, 1_G) : \mathbf{u} \in \mathrm{Sch}(G)\} \cup \{(\mathbf{u}, 1_G) : \mathbf{u} \text{ converges to } 1_G \text{ in } G\} \subset \Psi.$$

Main Theorem 1 (B., Zlatoš)

Let G be a topological group. Then $\mathfrak{b}G$ is topologically isomorphic to the quotient semigroup $\beta G/\Psi$.

The latter theorem implies the following.

Corollary (B., Zlatoš)

Let G be a discrete group. Then $\mathfrak{b}G$ is topologically isomorphic to the quotient semigroup $\beta G/\rho$, where ρ is the smallest closed congruence merging all Schur ultrafilters to the unit of G.

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Let G be a discrete group. Then $\mathfrak{b}G$ is topologically isomorphic to the quotient semigroup $\beta G/\rho$, where ρ is the smallest closed congruence merging all Schur ultrafilters to the unit of G.

Chart groups

Let G be a right topological group. By $\Lambda(G)$ we denote the set of all $g \in G$ such that the left shift $\lambda_g : G \to G$, $\lambda_g(x) = gx$ is continuous.

A right topological group G is called admissible if $\Lambda(G)$ is dense in G.

If G is a compact right topological group, then $\Lambda(G)$ is a subgroup of G.

A compact Hausdorff admissible right topological group is called a chart group.

Chart groups naturally appear in topological dynamics, as enveloping semigroups of dynamical systems.

One of the central problems in the theory of chart groups is when a chart group is a topological group.

Theorem

A chart group is a topological group if it has one of the following conditions:

- metrizable (Namioka);
- first-countable (Moors, Namioka);
- Fréchet-Urysohn (Glasner, Megrelishvili);
- countable tightness (Reznichenko).

We complemented the aforementioned result as follows:

Main Theorem 2 (B., Zlatoš)

For a chart group G the following conditions are equivalent:

- G is a topological group;
- every Schur ultrafilter on G converges to 1_G;
- there exists a dense subgroup $H \subseteq \Lambda(G)$ such that every Schur ultrafilter on H converges to 1_G.

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Proposition (B., Zlatoš)

Let G be a discrete group and θ be the least closed congruence on βG merging all idempotent ultrafilters to 1_G. Then $\beta G/\theta$ is a chart group with the following universal property: if H is a chart group and $f : G \to H$ is a continuous homomorphism such that $f(G) \subseteq \Lambda(H)$, then there exists a continuous homomorphism $\phi : \beta G/\theta \to H$ such that the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "\xi"] \arrow[d, "f"'] & \beta G/\theta \arrow[dl, "\phi"] \\ H & \end{tikzcd}$$

Final remarks

There exists a chart group which is not a topological group.

This example (not directly) implies the existence of a discrete group G and a closed congruence ρ on βG, which merges all idempotent ultrafilters to 1_G, but not all Schur ultrafilters. It also follows that our first main theorem is quite sharp.

Proposition (B., Zlatoš)

Let G be a chart group. Then each idempotent ultrafilter $\mathbf{u}$ such that $\Lambda(G) \in \mathbf{u}$ converges to 1_G.

So, the scale of the difference between chart groups and compact topological groups is the same (in some sense) as the scale of the difference between Schur and idempotent ultrafilters.

Thank You for attention!