Normed and Banach groups

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Definitions of normed and Banach groups

The theory of normed and Banach spaces is one of the most important and intensively studied branches of Mathematics. Taking also into account that it plays a fundamental role in almost all its areas, it seems reasonable and important to define and study classes of normed and Banach groups in the wider class of abelian topological groups. In the talk, based on my article [2], I propose very natural extensions of the notions of normed spaces and Banach spaces in the sense that a real locally convex space is a (Banach) normed space if and only if it is a (Banach) normed group.

Let E be a normed space whose topology is defined by a norm $\|\cdot\|$. Denote by $B := \{x \in E : \|x\| \leq 1\}$ the closed unit ball of E. It is clear that the sequence

$$\tfrac{1}{n} B = \left\{ x \in E : \|x\| \leq \tfrac{1}{n} \right\} \quad (n \in \mathbb{N})$$

defines a base at zero of the topology of E. Therefore the topology of E is defined by the operation of taking *homothetic images* of some *specific* subset of E, namely, the closed unit ball B of E. Observe that B has an important property of being an *absolutely convex* subset of E.

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Therefore, to define a natural analogue of normed spaces in the wider class of abelian topological groups we should take into account the following two conditions:

(1) there is $B \subseteq E$ such that the family $\{\frac{1}{n}B\}_{n\in\mathbb{N}}$ is a base of the topology of E,

(2) the set B in the condition (1) is absolutely convex.

It should be noticed that the conditions (1) and (2) are independent as the classical complete metrizable spaces ℓ_p with $p \in (0,1)$ show.

Let now (G,τ) be an abelian topological group, and let U be a subset of G. For every natural number $n \in \mathbb{N}$, put

$$U_{(n)} := \{g \in G : g, 2g, \dots, ng \in U\}.$$

Then, for the closed unit ball B of a normed space E, we have $\frac{1}{n}B = B_{(n)}$. Therefore to define the notion of a normed group G we should find $U \subseteq G$ such that the sequence $\{U_{(n)} : n \in \mathbb{N}\}$ is a base at zero in G. This is the first condition on U. The second one is that U should be a *quasi-convex* subset of G. The class of locally quasi-convex abelian groups was defined by Vilenkin in [3] and it contains all (real) locally convex spaces considered as abelian topological groups and all locally compact abelian (LCA) groups.

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Let (G,τ) be an abelian topological group, and let $\mathbb{S}_+ := \{z \in \mathbb{S} : \mathrm{Re}(z) \geq 0\}$. We denote by $\widehat{G}$ the group of all continuous characters (=homomorphisms from G into $\mathbb{S}$) of G. A subset A of G is called *quasi-convex* if for every $g \in G \setminus A$ there exists $\chi \in \widehat{G}$ such that $\chi(g) \notin \mathbb{S}_+$ and $\chi(A) \subseteq \mathbb{S}_+$. Following Vilenkin [3], (G,τ) is called *locally quasi-convex* (*lqc* for short) if it admits a neighborhood base at zero consisting of quasi-convex sets. Now we are ready to define

Definition 1: An abelian topological group (G,τ) is called a *normed group* if there exists a quasi-convex subset U of G such that the sequence $\{U_{(n)} : n \in \mathbb{N}\}$ is a base at zero in G.

By the Birkhoff–Kakutani metrization theorem, any normed group is metrizable. Since Banach spaces are *complete* normed spaces, we define the notion of a Banach group in a natural way.

Definition 2: A normed and complete group is called a *Banach group*.

Proposition 1: *A real locally convex space E is a normed (Banach) group if and only if it is a normed (Banach) space.*

Therefore the notions of normed and Banach groups indeed extend the notions of normed and Banach spaces, respectively.

Basic properties

Below we select several general properties.

Proposition 2: *Let H be a dense subgroup of a lqc abelian group G. If H is a normed group, then also G is a normed group. Consequently, the completion of a normed group is a Banch group.*

Proposition 3: *A subgroup of a normed group is a normed group.*

Proposition 4: *A closed subgroup of a Banach group is a Banach group.*

Proposition 5: *The class of normed (Banach) groups is closed under taking finite products.*

Let us recall that any LCA group is nuclear, and each nuclear group is a Schwartz group. It is well known that any Schwartz normed space is finite-dimensional. The next theorem essentially extends this result.

Theorem 1: *For an lqc abelian group G TFAE:*

(i) *G is a locally compact Banach (locally precompact normed) group;*

(ii) *G is a nuclear Banach (nuclear normed) group;*

(iii) *G is a Schwartz Banach (Schwartz normed) group;*

(iv) *there are $n, m \in \omega$ and a discrete abelian group D such that G is topologically isomorphic to (a dense subgroup of) $\mathbb{R}^n \times \mathbb{T}^m \times D$.*

Examples of Banach groups

Proposition 6: *Let K be a compact space. If G is a Banach (normed) group, then so is $C(K,G)$.*

Where $C(K,G)$ is the group of all continuous functions from K to G endowed with the compact-open topology whose base at zero is the family of sets

$$[K;U] := \{f \in C(K,G) : f(K) \subseteq U\}, \text{ where U is an open neighborhood of } 0 \in G$$

Proposition 7: *For every infinite cardinal κ, the quotient groups $\ell_\infty(\kappa)/\mathbb{Z}^\kappa$, $c_0(\kappa)/\mathbb{Z}^{(\kappa)}$ and $\ell_1(\kappa)/\mathbb{Z}^{(\kappa)}$ are Banach group.*

An analogous result for $1 < p < \infty$ is not true since, by [1], the group $\ell_p/\mathbb{Z}^{(\mathbb{N})}$ is not even locally quasi-convex.

Now we consider a totally different type of Banach groups. Recall that a sequence $\mathbf{u} = \{u_n\}$ in an abelian group G is a *T-sequence* if there is a (Hausdorff) group topology on G in which u_n converges to zero. The group G equipped with the finest group topology $\tau_{\mathbf{u}}$ with this property is denoted by $(G, \tau_{\mathbf{u}})$.

Proposition 8: *Let $\mathbf{u} = \{u_n\}$ be a T-sequence in an infinite abelian group G such that $G = \langle \mathbf{u} \rangle$. Then the dual group $(G, \tau_{\mathbf{u}})^\wedge$ is a Banach group.*

Representation theorems of Banach and normed groups

Below we give three representation of Banach and normed groups.

Theorem 2: *Let G be a normed (Banach) group. Then there is a set K such that the Banach space $\ell_\infty(K)$ contains a (closed) subgroup H and a closed subgroup Z such that $Z \subseteq H$ and the group G is topologically isomorphic to the quotient group H/Z.*

Theorem 3: *For every (separable) normed group G there is a (resp., metrizable) compact space K such that (G, τ) embeds into a (resp., separable) Banach group $C(K, \mathbb{S})$.*

Theorem 4: *Let G be a normed (Banach) group. Then there is a set K such that G embeds into the Banach group $\ell_\infty(K; \mathbb{S})$.*

Representation theorem for lqc abelian groups

Now we provide a representation theorem of locally quasi-convex abelian groups G. Recall that G is *reflexive* if the canonical embedding $G \mapsto G^{\wedge\wedge}$ is a topological isomorphism, where $G^{\wedge}$ denotes the dual group $\widehat{G}$ of G endowed with the compact-open topology.

Theorem 5: *An abelian topological group G is locally quasi-convex* (*and metrizable*) *if and only if it embeds into a reflexive* (*resp., metrizable*) *group.*

Almost normed and almost Banach groups

Now we introduce more general classes of groups.

Definition 3: An abelian topological group G is called an *almost normed group* if there is a compact subgroup K of G such that the quotient group G/K is normed. If in addition the group G is complete it is called an *almost Banach group*.

Proposition 9: (i) *The class of almost normed (Banach) groups is closed under taking finite products.*
(ii) *Let H be a dense subgroup of an lqc abelian group G. If H is an almost normed group then so is G. Consequently, the completion of an almost normed group is an almost Banach group.*
(iii) *A closed subgroup H of an almost normed (Banach) group G is an almost normed (Banach) group.*

Theorem 6: *An abelian topological group G is LCA if and only if it is a Schwartz, almost Banach group.*

Almost normed and almost Banach groups

Theorem 7: *For an abelian topological group G the following assertions are equivalent:*

(i) *G is a locally precompact, almost normed group;*

(ii) *G is a nuclear, almost normed group;*

(iii) *G is a Schwartz, almost normed group;*

(iv) *G contains a compact subgroup K such that G/K is topologically isomorphic to a dense subgroup of $\mathbb{R}^n \times \mathbb{T}^m \times D$, where $n, m \in \omega$ and D is discrete.*

We finish with the following problem:

Problem 1: Characterize (almost) Banach groups which are reflexive.

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Thank you!