Semi-topological properties of the K-topological version of the Jordan curve theorem and its applications

Sang-Eon Han,
Jeonbuk National University, Korea (Republic of)
sehan@jbnu.ac.kr
Homepage:https://sehan.jbnu.ac.kr

July 8-12, 2024

Papers related to this work

(1) S.-E. Han, Covering rough set structures for a locally finite covering approximation space, *Information Sciences* **480**(2019) 420-437.
(2) S.-E. Han, Roughness measures of locally finite covering rough sets, *Int. Jour. Approximate Reasoning.* **105**(2019) 368-385.
(3) S.-E. Han, Jordan surface theorem for simple closed *SST*-surfaces, *Topol. Appl.* **272**(2020) 106953.
(4) S.-E. Han, Digital topological rough set structures and topological operators, *Topol. Appl.* **301**(2021) 107507.
(5) S.-E. Han, Wei Yao, Semi-topological properties of the *K*-topological version of the Jordan curve theorem, *Results in Mathematics*, 79(7) (2024), 1-20.

Aims of the talk

Let C_K^l be a simple closed Khalimsky (K-, for brevity) curve with l-elements in $(\mathbb{Z}^2, \kappa^2)$.

• Investigation of the semi-topological properties of the Khalimsky (K-, for brevity) topological version of the classical Jordan curve theorem.

On the K-topological plane, i.e., $(\mathbb{Z}^2, \kappa^2)$, consider a simple closed K-curve with l elements, denoted by C_K^l. Then

• every C_K^l separates $(\mathbb{Z}^2, \kappa^2)$ into exactly two nonempty components that may be neither open nor closed in $(\mathbb{Z}^2, \kappa^2)$.

Based on these features, we need to investigate semi-topological features of C_K^l and $\mathbb{Z}^2 \setminus C_K^l$ in $(\mathbb{Z}^2, \kappa^2)$.

We first show that

• Not every C_K^l is always semi-open or semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

• We find a condition for C_K^l to be either semi-open or semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

• After establishing a continuous analog of C_K^l denoted by $\mathcal{A}(C_K^l)\,(\subset \mathbb{R}^2)$, we show that $\mathcal{A}(C_K^l)$ is both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$ that is the 2-dimensional real plane with the usual topology.

• We show that $\mathcal{A}(C_K^l)$ always separates $(\mathbb{R}^2, \mathcal{U})$ into two non-empty components, denoted by C and D, that are both semi-open and semi-closed to obtain a partition of $\mathbb{R}^2$, i.e., $\{C, D, \mathcal{A}(C_K^l)\}$.

• Finally, given C_K^l, we obtain a partition of $\mathbb{Z}^2$, i.e., $\{I(C_K^l), O(C_K^l), C_K^l\}$ and prove that each of $I(C_K^l)$ and $O(C_K^l)$ is semi-closed and it need not be semi-open, where $I(C_K^l)$ and $O(C_K^l)$ are called an inside and outside of C_K^l, respectively.

Essential notions-1

- For $a, b \in \mathbb{Z}$, $[a, b]_{\mathbb{Z}} := \{t \in \mathbb{Z} \,|\, a \leq t \leq b\}$.
- n-dimensional K-topological space, $n \geq 1$:$(\mathbb{Z}^n, \kappa^n)$

Khalimsky line topology κ on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set

$$\{[2n-1, 2n+1]_{\mathbb{Z}} \,|\, n \in \mathbb{Z}\} \text{ as a subbase.}$$

Furthermore, the product topology on $\mathbb{Z}^n$ induced by $(\mathbb{Z}, \kappa)$ is called the n-dimensional K-topological space, denoted by $(\mathbb{Z}^n, \kappa^n)$.

- For a point $p := (x, y) \in \mathbb{Z}^2$, we take the notations

$N_4(p) := \{(x \pm 1, y), p, (x, y \pm 1)\}$,
$N_8(p) := \{(x \pm 1, y), p, (x, y \pm 1), (x \pm 1, y \pm 1)$.

Essential notions-2

Let us now recall some structures of $(\mathbb{Z}^n, \kappa^n)$.

• A point $x = (x_i)_{i \in [1,n]_{\mathbb{Z}}} \in \mathbb{Z}^n$ is *pure open* if all coordinates are odd, and *pure closed* if each of the coordinates is even and the other points in $\mathbb{Z}^n$ are called *mixed*. These points are shown like the following symbols: The symbol ■ (*resp.* •) means a pure closed point (*resp.* a mixed point) (see Figure 1) and further, a black jumbo dot represents a pure open point.

• $(\mathbb{Z}^n)_o$ (*resp.* $(\mathbb{Z}^n)_e$) indicates the set of all pure open (*resp.* pure closed) points of $(\mathbb{Z}^n, \kappa^n)$.

• $(\mathbb{Z}^n)_m = \mathbb{Z}^n \setminus ((\mathbb{Z}^n)_e \cup (\mathbb{Z}^n)_o)$ stands for the set of all mixed points of $(\mathbb{Z}^n, \kappa^n)$, $n \in \mathbb{N} \setminus \{1\}$.

Essential notions-3

• For $X \subset \mathbb{Z}^n$, $SN_K(p)$:= the smallest open set of the point p in (X, κ_X^n). Besides, in (X, κ_X^n) we will take the notation $Cl(\{p\})$ for the closure of the given singleton $\{p\}$.

• Given a topological space (X, T), a subset A of (X, T) is semi-open if and only if $A \subset Cl(Int(A))$ (Levine (1963)) and a subset B of (X, T) is semi-closed if and only if $Int(Cl(B)) \subset B$ (Crosseley et al. (1978, 1971)), where "*Int*" means the interior of the given set B under (X, T). Hence it is clear that each of an empty set and the total set is both semi-open and semi-closed.

Definition 1

We say that a simple closed K-curve with l elements in $(\mathbb{Z}^2, \kappa^2), l \in \mathbb{N}_e \setminus \{2, 6\}$, denoted by $C_K^{2,l}, l \geq 4$, is a finite sequence $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}} \subset \mathbb{Z}^2$ such that x_i and x_j are K-adjacent if and only if $\mid i - j \mid = \pm 1 (mod\ l)$ (see Figures 1, 2, 5, and 8), where we say that x_i and x_j are K-adjacent if $x_i \in SN_K(x_j)$ or $x_j \in SN_K(x_i)$.

Figure 1

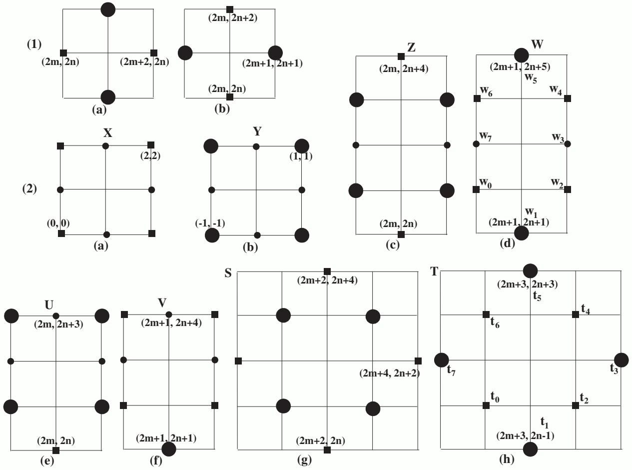

Figure: (1) Each of (a) and (b) is C_K^4. (2) Configuration of the several kinds of C_K^8 in (a)-(h) as examples.

Figure 2

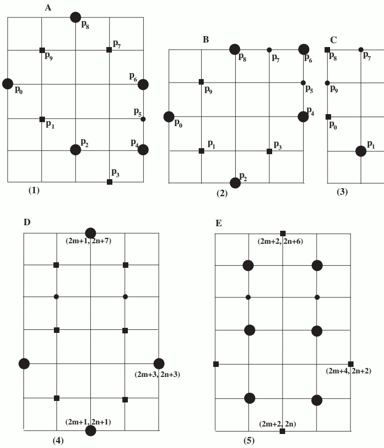

Figure: Configuration of several types of C_K^{10} as examples. Furthermore, A is not semi-closed but semi-open (see (1)). B is both semi-closed and semi-open (see (2)). C is neither semi-open nor semi-closed (see (3)). D

Some properties of simple closed K-curves with respect to the semi-openness and semi-closedness

Lemma 3.1

(Han et al.(2023)) and Nada (2004)) In $(\mathbb{Z}^n, \kappa^n)$, a non-empty set $X(\subset \mathbb{Z}^n)$ is semi-open if and only if for each $x \in X$, $SN_K(x) \cap X_{op} \neq \emptyset$.

Using Lemma 3.1 and the notions of semi-open and semi-closed, we obtain the following (see Lemma 3.2).

Lemma 3.1

(Han et al. (2023)) In $(\mathbb{Z}^n, \kappa^n)$, $X(\subset \mathbb{Z}^n)$ is semi-closed if and only if for each $y \in \mathbb{Z}^n \setminus X$, $SN_K(y) \cap (\mathbb{Z}^n \setminus X)_{op} \neq \emptyset$.

Some properties of simple closed K-curves with respect to the semi-openness and semi-closedness

Lemma 3.1

(Han et al.(2023)) and Nada (2004)) In $(\mathbb{Z}^n, \kappa^n)$, a non-empty set $X(\subset \mathbb{Z}^n)$ is semi-open if and only if for each $x \in X$, $SN_K(x) \cap X_{op} \neq \emptyset$.

Using Lemma 3.1 and the notions of semi-open and semi-closed, we obtain the following (see Lemma 3.2).

Lemma 3.1

(Han et al. (2023)) In $(\mathbb{Z}^n, \kappa^n)$, $X(\subset \mathbb{Z}^n)$ is semi-closed if and only if for each $y \in \mathbb{Z}^n \setminus X$, $SN_K(y) \cap (\mathbb{Z}^n \setminus X)_{op} \neq \emptyset$.

Theorem 3.2 (Results in Math, 2024, Han)

Let $C_K^l := (c_i)_{i \in [0, l-1]_{\mathbb{Z}}}$ with the following condition (1) or (2).
(1) There is a subsequence

$$X_1 := \{c_{t-1(mod\ l)}, c_t, c_{t+1(mod\ l)}\} \tag{3.2}$$

of C_K^l such that
(1-1) each of $c_{t-1(mod\ l)}$ and $c_{t+1(mod\ l)}$ is a pure open point,
(1-2) c_t is a pure closed point, and
(1-3) there is a mixed point $c \in \mathbb{Z}^2 \setminus C_K^l$ such that

$$\{c_{t-1(mod\ l)}, c_{t+1(mod\ l)}\} \subset SN_K(c).$$

Theorem 3.2-continued (Results in Math, 2024, Han)

(2) There is a subsequence

$$\begin{cases} X_2 := \{c_{t-3(mod\, l)}, c_{t-2(mod\, l)}, c_{t-1(mod\, l)}, c_t, \\ c_{t+1(mod\, l)}, c_{t+2(mod\, l)}, c_{t+3(mod\, l)}\} \end{cases} \quad (3.3)$$

of C_K^l such that

$$\begin{cases} A := \{c_{t-3(mod\, l)}, c_{t-1(mod\, l)}, c_{t+1(mod\, l)}, \\ c_{t+3(mod\, l)}\} \subset (\mathbb{Z}^2)_o, A \subset N_8(c), \\ \text{where } c \in \mathbb{Z}^2 \setminus C_K^l \text{ and } c \in (\mathbb{Z}^2)_e. \end{cases} \quad (3.4)$$

Then C_K^l is not semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Example 3.5, (2024)

(1) Consider the C_K^4 in Figure 1(1)(a) and (b). Based on the cases, by Lemmas 3.1 and 3.2, it is clear that each C_K^4 in Figure 1(1) is not semi-closed but only semi-open in $(\mathbb{Z}^2, \kappa^2)$ (see the condition (1) of Theorem 3.6).

(2) Consider the $C_K^8 := (y_i)_{i \in [0,7]_{\mathbb{Z}}}$ in Figure 1(2)(b). Then, owing to the given four points

$y_0 := (1,1), y_2 := (-1,1), y_4 := (-1,-1), y_6 := (1,-1)$ in C_K^8 as in Figure 1(2)(b), it is not semi-closed but semi-open in $(\mathbb{Z}^2, \kappa^2)$ (see the condition (1) of Theorem 3.6).

Figure 3

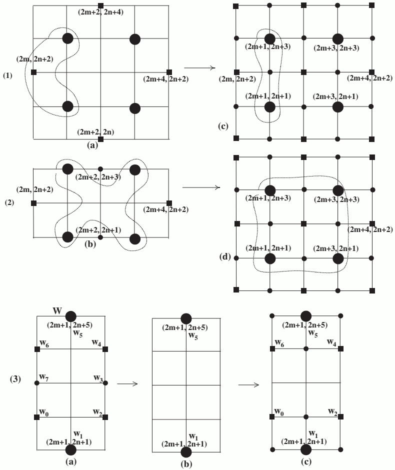

Figure: (1) and (2) The process of being the non-semi-closedness of the given C_K^8. (3) The proof of the non-semi-openness of the given C_K^8.

Theorem 3.6, Results in Math, 2024,Han

There are three types of C_K^l according to the number l of C_K^l, as follows:
($*1$) Either semi-open (not semi-closed) or semi-closed (not semi-open);
($*2$) Both semi-open and semi-closed; or
($*3$) Neither semi-open nor semi-closed.
To be precise,
(1) in the case of $l = 4$, C_K^4 is semi-open (not semi-closed),
(2) in the case of $l = 8$, C_K^8 appears with one of the types of ($*1$) and ($*2$), and
(3) in the case of $l \notin \{4, 8\}$, the semi-openness and semi-closedness of C_K^l depends on the situation.
Note that C_K^l may be neither semi-open nor semi-closed in $(\mathbb{Z}^2, \kappa^2)$ if $10 \leqslant l \in \mathbb{N}_e$, i.e., $l \notin \{4, 8\}$.

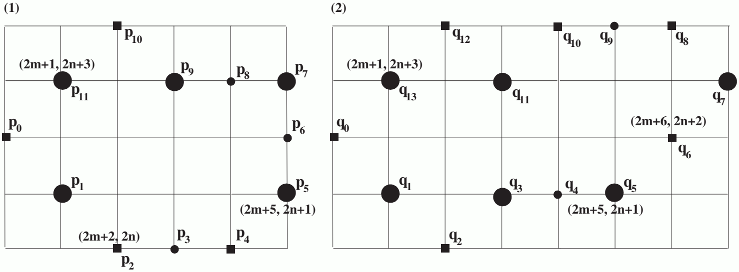

Figure: As mentioned in Theorem 3.6, in the case of C_K^l with $l \in [10, \infty)_{\mathbb{Z}} \cap \mathbb{N}_e$, the semi-openness and semi-closedness of C_K^l depends on the situation, as follows: (1) Configuration of both the non-semi-openness and non-semi-closedness of the given C_K^{12}. (2) Configuration of both the non-semi-openness and non-semi-openness of the given C_K^{14}.

Proposition 3.7 (Results in Math, 2024, Han)

C_K^l is not semi-closed in $(\mathbb{Z}^2, \kappa^2)$ if and only if it has a subset X_1 of (3.2) satisfying the properties of (1-1)–(1-3) of Theorem 3.2 or $X_2(\subset C_K^l)$ of (3.3) satisfying the properties of (3.4) of Theorem 3.2.

Thoerem 3.8

A subset $X \subset \mathbb{Z}^2$ is not semi-closed in $(\mathbb{Z}^2, \kappa^2)$ if and only if there is a point $p \in \mathbb{Z}^2 \setminus X$ such that

$$\left\{\begin{array}{l}(1)\, p \in (\mathbb{Z}^2)_m \text{ and } N_4(p) \cap (\mathbb{Z}^2)_o \subset X \text{ or} \\ (2)\, p \in (\mathbb{Z}^2)_e \text{ and } N_8(p) \setminus N_4(p) \subset X.\end{array}\right\} \quad (3.6)$$

Establishment of an operator transforming a subspace of $(\mathbb{Z}^2, \kappa^2)$ into a subspace of $(\mathbb{R}^2, \mathcal{U})$

Definition 2, (Comput. Appl. Math., 2017, Han)

For $p := (p_1, p_2) \in (\mathbb{Z}^2, \kappa^2)$, the continuous analog of the point p, denoted by K_p, is defined as follows (see Figure 6);

$$\left\{\begin{array}{l}\{(t_1, t_2) \,|\, t_i \in [2x_i - 0.5, 2x_i + 0.5], i \in \{1, 2\}\},\\ \text{if } p = (2x_1, 2x_2);\\ \{(t_1, t_2) \,|\, t_i \in (2x_i + 0.5, 2x_i + 1.5), i \in \{1, 2\}\},\\ \text{if } p = (2x_1 + 1, 2x_2 + 1);\\ \{(t_1, t_2) \,|\, t_1 \in (2x_1 + 0.5, 2x_1 + 1.5), t_2 \in [2x_2 - 0.5, 2x_2 + 0.5]\},\\ \text{if } p = (2x_1 + 1, 2x_2); \text{ and}\\ \{(t_1, t_2) \,|\, t_1 \in [2x_1 - 0.5, 2x_1 + 0.5], t_2 \in (2x_2 + 0.5, 2x_2 + 1.5)\},\\ \text{if } p = (2x_1, 2x_2 + 1).\end{array}\right\} \quad (4.1)$$

Figure 6

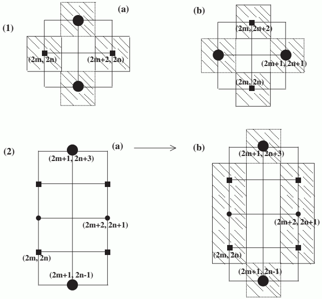

Figure: Configuration of an $\mathcal{A}(C_K^4)$ (see (1)) and an $\mathcal{A}(C_K^8)$ (see the process from (a) to (b) in (2)) that are both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$.

Definition 3 (Results in Math, 2024, Han)

For $X \subset \mathbb{Z}^2$, using $K_p, p \in X \subset \mathbb{Z}^2$, an operator (or continuous analog) is defined as

$$\left\{ \begin{array}{l} \mathcal{A} : P(\mathbb{Z}^2) \to P(\mathbb{R}^2) \text{ such that} \\ \mathcal{A}(X) = \bigcup_{p \in X} K_p \subset \mathbb{R}^2, \text{ e.g.} \\ \mathcal{A}(\mathbb{Z}^2) = \bigcup_{p \in \mathbb{Z}^2} K_p = \mathbb{R}^2. \end{array} \right\} \quad (4.2)$$

In (4.2), $P(T)$ means the power set of the given set T, where $T \in \{\mathbb{Z}^2, \mathbb{R}^2\}$.
Then we assume $\mathcal{A}(X)$ to be $(\mathcal{A}(X), \mathcal{U}_{\mathcal{A}(X)})$ that is a subspace of the usual topology $(\mathbb{R}^2, \mathcal{U})$.

Lemma 4.1

For distinct points $x_1, x_2 \in X \subset (\mathbb{Z}^2, \kappa^2)$, we obtain the following:
(1) If x_1 is K-adjacent to x_2, then the set $K_{x_1} \cup K_{x_2}$ is connected in $(\mathbb{R}^2, \mathcal{U})$. The converse also holds.
(2) If x_1 is not K-adjacent to x_2, then the set $K_{x_1} \cup K_{x_2}$ is disconnected in $(\mathbb{R}^2, \mathcal{U})$. The converse also holds.

Lemma 4.4

The operator $\mathcal{A}$ of (4.2) preserves both the connectedness and disconnectedness.

Theorem 4.5

For any C_K^l, $\mathcal{A}(C_K^l)$ is both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$.

Lemma 4.1

For distinct points $x_1, x_2 \in X \subset (\mathbb{Z}^2, \kappa^2)$, we obtain the following:
(1) If x_1 is K-adjacent to x_2, then the set $K_{x_1} \cup K_{x_2}$ is connected in $(\mathbb{R}^2, \mathcal{U})$. The converse also holds.
(2) If x_1 is not K-adjacent to x_2, then the set $K_{x_1} \cup K_{x_2}$ is disconnected in $(\mathbb{R}^2, \mathcal{U})$. The converse also holds.

Lemma 4.4

The operator $\mathcal{A}$ of (4.2) preserves both the connectedness and disconnectedness.

Theorem 4.5

For any C_K^l, $\mathcal{A}(C_K^l)$ is both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$.

Lemma 4.1

For distinct points $x_1, x_2 \in X \subset (\mathbb{Z}^2, \kappa^2)$, we obtain the following:
(1) If x_1 is K-adjacent to x_2, then the set $K_{x_1} \cup K_{x_2}$ is connected in $(\mathbb{R}^2, \mathcal{U})$. The converse also holds.
(2) If x_1 is not K-adjacent to x_2, then the set $K_{x_1} \cup K_{x_2}$ is disconnected in $(\mathbb{R}^2, \mathcal{U})$. The converse also holds.

Lemma 4.4

The operator $\mathcal{A}$ of (4.2) preserves both the connectedness and disconnectedness.

Theorem 4.5

For any C_K^l, $\mathcal{A}(C_K^l)$ is both semi-open and semi-closed in $(\mathbb{R}^2, \mathcal{U})$.

Proposition 4.6

$\mathcal{A}(C_K^l)$ separates $(\mathbb{R}^2, \mathcal{U})$ into exactly two non-empty components that are both semi-open and semi-closed. Namely, there is a partition $\{\mathcal{A}(C_K^l), C, D\}$ of $\mathbb{R}^2$, where C and D are non-empty components of $\mathbb{R}^2 \setminus \mathcal{A}(C_K^l)$ and one of them is bounded and the other is unbounded under the Euclidean metric.

Semi-topological properties of the K-topological version of the Jordan curve theorem

Remark 5.1

While C_K^l separates $(\mathbb{Z}^2, \kappa^2)$ into exactly two components (Kiselman (2000) and Šlapal (2006, 2008)), each of the two components is neither closed nor open in $(\mathbb{Z}^2, \kappa^2)$.
For instance, consider the set $Z := C_K^8$ in Figure 1(2)(c). Then the finite component of $\mathbb{Z}^2 \setminus Z$ may be neither closed nor open in $(\mathbb{Z}^2, \kappa^2)$. Besides, the infinite component of $\mathbb{Z}^2 \setminus Z$ is neither closed nor open in $(\mathbb{Z}^2, \kappa^2)$.

Definition 4 (Results in Math, 2024, Han)

Given a C_K^l, consider $\mathcal{A}(C_K^l)$. Then, we define the following two notions.
(1) $I(C_K^l) := B(\mathbb{R}^2 \setminus \mathcal{A}(C_K^l)) \cap \mathbb{Z}^2$, where $B(\mathbb{R}^2 \setminus \mathcal{A}(C_K^l))$ means the bounded component of $\mathbb{R}^2 \setminus \mathcal{A}(C_K^l)$

Semi-topological properties of the K-topological version of the Jordan curve theorem

Remark 5.1

While C_K^l separates $(\mathbb{Z}^2, \kappa^2)$ into exactly two components (Kiselman (2000) and Šlapal (2006, 2008)), each of the two components is neither closed nor open in $(\mathbb{Z}^2, \kappa^2)$.
For instance, consider the set $Z := C_K^8$ in Figure 1(2)(c). Then the finite component of $\mathbb{Z}^2 \setminus Z$ may be neither closed nor open in $(\mathbb{Z}^2, \kappa^2)$. Besides, the infinite component of $\mathbb{Z}^2 \setminus Z$ is neither closed nor open in $(\mathbb{Z}^2, \kappa^2)$.

Definition 4 (Results in Math, 2024, Han)

Given a C_K^l, consider $\mathcal{A}(C_K^l)$. Then, we define the following two notions.
(1) $I(C_K^l) := B(\mathbb{R}^2 \setminus \mathcal{A}(C_K^l)) \cap \mathbb{Z}^2$, where $B(\mathbb{R}^2 \setminus \mathcal{A}(C_K^l))$ means the bounded component of $\mathbb{R}^2 \setminus \mathcal{A}(C_K^l)$

Remark 5.2

(1) Both of $I(C_K^l)$ and $O(C_K^l)$ are connected subsets in $(\mathbb{Z}^2, \kappa^2)$.
(2) $I(C_K^l) \cap O(C_K^l) = \emptyset$.
(3) Each of $I(C_K^l)$ and $O(C_K^l)$ is connected with C_K^l.

Lemma 5.3

$I(C_K^4)$ is not semi-open but it is semi-closed.

Lemma 5.4

In the case of $l = 8$ (see Figure 1(2)(a)–(h)), we consider the following: For the objects X, Y, Z, S, T, U, V, W in $(\mathbb{Z}^2, \kappa^2)$ in Figure 1(2), we obtain the following.
(1) Each of $I(X)$, $I(V)$, $I(W)$, and $I(T)$ is both semi-open and semi-closed in $(\mathbb{Z}^2, \kappa^2)$.
(2) Each of $I(Y)$, $I(Z)$, $I(U)$, and $I(S)$ is not semi-open but semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Remark 5.2

(1) Both of $I(C_K^l)$ and $O(C_K^l)$ are connected subsets in $(\mathbb{Z}^2, \kappa^2)$.
(2) $I(C_K^l) \cap O(C_K^l) = \emptyset$.
(3) Each of $I(C_K^l)$ and $O(C_K^l)$ is connected with C_K^l.

Lemma 5.3

$I(C_K^4)$ is not semi-open but it is semi-closed.

Lemma 5.4

In the case of $l = 8$ (see Figure 1(2)(a)–(h)), we consider the following: For the objects X, Y, Z, S, T, U, V, W in $(\mathbb{Z}^2, \kappa^2)$ in Figure 1(2), we obtain the following.
(1) Each of $I(X)$, $I(V)$, $I(W)$, and $I(T)$ is both semi-open and semi-closed in $(\mathbb{Z}^2, \kappa^2)$.
(2) Each of $I(Y)$, $I(Z)$, $I(U)$, and $I(S)$ is not semi-open but semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Remark 5.2

(1) Both of $I(C_K^l)$ and $O(C_K^l)$ are connected subsets in $(\mathbb{Z}^2, \kappa^2)$.
(2) $I(C_K^l) \cap O(C_K^l) = \emptyset$.
(3) Each of $I(C_K^l)$ and $O(C_K^l)$ is connected with C_K^l.

Lemma 5.3

$I(C_K^4)$ is not semi-open but it is semi-closed.

Lemma 5.4

In the case of $l = 8$ (see Figure 1(2)(a)–(h)), we consider the following: For the objects X, Y, Z, S, T, U, V, W in $(\mathbb{Z}^2, \kappa^2)$ in Figure 1(2), we obtain the following.
(1) Each of $I(X)$, $I(V)$, $I(W)$, and $I(T)$ is both semi-open and semi-closed in $(\mathbb{Z}^2, \kappa^2)$.
(2) Each of $I(Y)$, $I(Z)$, $I(U)$, and $I(S)$ is not semi-open but semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Lemma 5.5

(1) $I(C_K^4)$ is not semi-open but it is semi-closed.
(2) In the case of $l = 8$ (see Figure 1(2)(a)–(h)), we consider the following: For the objects X, Y, Z, S, T, U, V, W in $(\mathbb{Z}^2, \kappa^2)$ in Figure 1(2), we obtain the following.
(2-1) Each of $I(X)$, $I(V)$, $I(W)$, and $I(T)$ is both semi-open and semi-closed in $(\mathbb{Z}^2, \kappa^2)$.
(2-2) Each of $I(Y)$, $I(Z)$, $I(U)$, and $I(S)$ is not semi-open but semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Motivated by Lemma 5.3, 5.4, and 5.5, we obtain the following:

Proposition 5.6 (Results in Math, 2024, Han)

Given C_K^l, $I(C_K^l)$ is semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Lemma 5.7

(1) For any $l \in \{4, 8, 10\}$, $O(C_K^l)$ is both semi-open and semi-closed.
(2) If $12 \leqslant l \in \mathbb{N}_e$, then $O(C_K^l)$ need not be semi-open but it is semi-closed.

Motivated by Lemma 5.3, 5.4, and 5.5, we obtain the following:

Proposition 5.6 (Results in Math, 2024, Han)

Given C_K^l, $I(C_K^l)$ is semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Lemma 5.7

(1) For any $l \in \{4, 8, 10\}$, $O(C_K^l)$ is both semi-open and semi-closed.
(2) If $12 \leqslant l \in \mathbb{N}_e$, then $O(C_K^l)$ need not be semi-open but it is semi-closed.

Figure 8

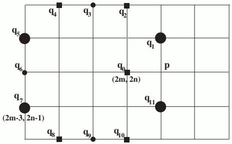

Figure: Given C_K^{12}, we obtain that $O(C_K^{12})$ is not semi-open in $(\mathbb{Z}^2, \kappa^2)$ (see the point p).

Theorem 5.8 (Results in Math, 2024, Han)

In $(\mathbb{Z}^2, \kappa^2)$, given C_K^l, we have the partition $\{C_K^l, I(C_K^l), O(C_K^l)\}$ such that each of $I(C_K^l)$ and $O(C_K^l)$ is semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Proof.

By Proposition 5.6 and Lemma 5.8, the proof is completed. □

Theorem 5.8 (Results in Math, 2024, Han)

In $(\mathbb{Z}^2, \kappa^2)$, given C_K^l, we have the partition $\{C_K^l, I(C_K^l), O(C_K^l)\}$ such that each of $I(C_K^l)$ and $O(C_K^l)$ is semi-closed in $(\mathbb{Z}^2, \kappa^2)$.

Proof.

By Proposition 5.6 and Lemma 5.8, the proof is completed. □

Thanks for your attention!