On quasi b-metric spaces, quasi-metrizability and bicompletions

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Outline

- Basic properties of quasi b-metric spaces
- Quasi metrizability of quasi b-metric spaces
- Symmetric completions of quasi b-metric spaces (extensions of quasi b-metric spaces)
- Characterizations of bicomplete quasi b-metric spaces

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Now recall

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Czerwik, 1993 Definition

Let X be a nonempty set. A map $p : X \times X \to [0, \infty)$ is a **b-metric** on X if for all $x, y, z \in X$ the following conditions hold:

- (i) $p(x, y) = 0 \Leftrightarrow x = y$,
- (ii) $p(x, y) = p(y, x)$,
- (iii) $p(x, z) \leq \alpha[p(x, y) + p(y, z)], \alpha \geq 1$.

The pair (X, α, p) is then called a **b-metric space**.

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Basic properties of quasi b-metric spaces

Recall [Shah M. H. et al, 2012]

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The pair (X, α, p) is then called a **quasi b-metric space**.

Note that for a quasi b-metric space (X, α, p) a mapping p is not continuous in both variables, unlike a metric or a quasi metric on a set.

Proposition

Let (X, α, p) be a quasi b-metric space, then (X, α, p^{-1}) is a quasi b-metric space, with $p^{-1}(x, y) = p(y, x)$, for $x, y \in X$.

Definition

We refer to p^{-1} as the **conjugate** of p. Note also, that (X, α, p^s) is a b-metric space with $p^s = p \vee p^{-1}$. So we have a **bispace** (X, p, p^{-1}). We say that a quasi b-metric space (X, α, p) is bicomplete, when the b-metric space (X, α, p^s) is complete.

Proposition

Let (X, α, p) be a quasi b-metric space. Then (X, α, p) is bicomplete if and only if (X, α, p^{-1}) is bicomplete.

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Let (X, α, p) be a quasi b-metric space. Then (X, α, p) is a b-metric space if and only if $p = p^{-1}$.

Definition

Let (X, α, p) be a quasi b-metric space, then the topology $\tau(p)$ has as a base the set $\{B_p(x, \epsilon)\}$, where $B_p(x, \epsilon)$ is an **ball** (not necessarily open in the usual sense) with centre x and radius $\epsilon > 0$. That is, $B_p(x, \epsilon) = \{y \in X : p(x, y) < \epsilon\}$ for each $x \in X$. For a subset A of X, we say A is open if for each $a \in A$, there exists $\epsilon > 0$, such that $B_p(a, \epsilon) \subseteq A$, and the topology generated by p on X is denoted by $\tau(p)$. So we obtain a **bitopological** space $(X, \tau(p), \tau(p^{-1}))$. In case $\alpha = 1$, then $(X, 1, p)$ is a T_0-quasi metric space.

Example

Let $X = \{0, 1, \frac{1}{2}, ..., \frac{1}{n}, ...\}$ and define $p : X \times X \to [0, \infty)$ by

$$p(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ 1 & \text{if} \quad x \neq y \in \{0,\ 1\}; \\ \max\{0,\ y - x\} & \text{if} \quad x \neq y \in \{0\} \cup \{\frac{1}{2n}, \geq 1\}; \\ 4 & \text{Otherwise.} \end{cases}$$

Then (X, α, p) is a quasi b-metric space with $\alpha = 4$. Note that (X, α, p) is not a b-metric space since $p(\frac{1}{2}, \frac{1}{6}) = 0$ and $p(\frac{1}{6}, \frac{1}{2}) = \frac{1}{3}$, so $p(\frac{1}{2}, \frac{1}{6}) \neq p(\frac{1}{6}, \frac{1}{2})$.

The conjugate $p^{-1} : X \times X \to [0, \infty)$ of p is defined by $p^{-1}(x, y)$ for all $x, y \in X$ is

$$p(y,x) = \begin{cases} 0 & \text{if} \quad y = x; \\ 1 & \text{if} \quad y \neq x \in \{0,\ 1\}; \\ \max\{0,\ x - y\} & \text{if} \quad y \neq x \in \{0\} \cup \{\frac{1}{2n}, \geq 1\}; \\ 4 & \text{Otherwise.} \end{cases}$$

Also, we get $p^s : X \times X \to [0, \infty)$ defined by

$$p^s(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ 1 & \text{if} \quad x \neq y \in \{0, 1\}; \\ |x - y| & \text{if} \quad x \neq y \in \{0\} \cup \{\frac{1}{2n}, \geq 1\}; \\ 4 & \text{Otherwise.} \end{cases}$$

So (X, α, p^s) is a b-metric space, with $\alpha = 4$.

Proposition

Let (X, α, p) be a quasi b-metric space. There exists a **bounded** quasi b-metric space (X, α, d) such that, $\tau(p) = \tau(d)$, and (X, α, p) is bicomplete if and only if (X, α, d) is bicomplete.

Corollary

Let (X, α, p) be a b-metric space. There exists a **bounded** b-metric space (X, α, d) such that, $\tau(p) = \tau(d)$, and (X, α, p) is complete if and only if (X, α, d) is complete.

Proposition

Let (X_i, α_i, p_i) be a finite sequence of quasi b-metric spaces, for $i = 1, 2, ..., n$. Put $X = X_1 \times X_2 \times ... \times X_n$, and $\alpha = max\{\alpha_i\}_{i=1}$. Define $p : X \times X \to [0, \infty)$ by

$$p(x, y) = \Sigma_{i=1}^{n} \frac{p_i(x_i, y_i)}{2^n},$$

with $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in X. Then (X, α, p) is a quasi b-metric space. Furthermore (X, α, p) is bicomplete if and only if (X_i, α_i, p_i) is bicomplete for each $i = 1, 2, ..., n$.

Corollary

Let (X_i, α_i, p_i) be a finite sequence of b-metric spaces, for $i = 1, 2, ..., n$. Put $X = X_1 \times X_2 \times ... \times X_n$, and $\alpha = max\{\alpha_i\}_{i=1}$. Define $p : X \times X \to [0, \infty)$ by

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Quasi-metrizability of quasi b-metric spaces

We recall that a topological space (X, τ) is **quasi metrizable** if there is a quasi-metric d on X such that $\tau = \tau(d)$. Given a quasi b-metric space $(X.\alpha, p)$ the topology $\tau(p)$ can canonically be defined by using quasi-uniform structures, it can easily be realised that the quasi-uniform space induced by p has a countable base hence $\tau(p)$ is quasi metrizable.

Theorem

For every quasi b-metric space (X, α, p), the topological space $(X, \tau(p))$, is quasi metrizable.

We revisit the above Theorem and construct a quasi metric d on the quasi b-metric space (X, α, p) directly, without appealing to quasi-uniformities.

Theorem

Let (X, α, p) be a quasi b-metric space, with $0 < m \leq 1$ such that $(2\alpha)^m = 2$ and for all $x, y \in X$, define $d : X \times X \to [0, \infty)$ by

$$d(x,y) = inf\{\Sigma_{i=1}^{n} p^m(x_i, x_{i+1}) : x_1 = x, x_2, , ..., x_{n+1} = y \in X\},$$

for $n \geq 1$. Then d is a quasi-metric on X satisfying

$$\frac{1}{4} p^m(x,y) \leq d(x,y) \leq p^m(x,y)$$

for all $x, y \in X$. In particular if p is a quasi-metric on X, then $p = d$. Also if the quasi b-metric space (X, α, p) is bicomplete, then the quasi metric space (X, d) is bicomplete.

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Let (X, α, p) be a quasi b-metric space, with $0 < m \leq 1$ such that $(2\alpha)^m = 2$ and for all $x, y \in X$, define $d^s : X \times X \to [0, \infty)$ by

$$d^s(x,y) = inf\{\Sigma_{i=1}^{n}(p^s)^m(x_i, x_{i+1}) : x_1 = x, x_2, , ..., x_{n+1} = y \in X\},$$

for $n \geq 1$. Then d^s is a metric on X satisfying

$$\frac{1}{4}(p^s)^m(x,y) \leq d^s(x,y) \leq (p^s)^m(x,y)$$

for all $x, y \in X$.

Paluszynski, Stempak, 2009 Theorem

Let (X, α, p) be a b-metric space, with $0 < m \leq 1$ such that $(2\alpha)^m = 2$ and for all $x, y \in X$, define $d : X \times X \to [0, \infty)$ by

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Using the previous result, take $m = \frac{1}{3}$, we construct a quasi metric $d : X \times X \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ \left(\frac{1}{4}\right)^{\frac{1}{3}} & \text{if} \quad x \neq y \in \{0,\ 1\}; \\ \max\{0,\ (y-x)^{\frac{1}{3}}\} & \text{if} \quad x \neq y \in \{0\} \cup \{\frac{1}{2n}, \geq 1\}; \\ \left(\frac{1}{4}\right)^{\frac{1}{3}} & \text{Otherwise.} \end{cases}$$

The conjugate of d is $d^{-1} : X \times X \to [0, \infty)$ given by

$$d^{-1}(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ (\frac{1}{4})^{\frac{1}{3}} & \text{if} \quad x \neq y \in \{0,\ 1\}; \\ \max\{0,\ (x-y)^{\frac{1}{3}}\} & \text{if} \quad x \neq y \in \{0\} \cup \{\frac{1}{2n}, \geq 1\}; \\ (\frac{1}{4})^{\frac{1}{3}} & \text{Otherwise.} \end{cases}$$

In this example the metric $d^s = d \vee d^{-1}$ is $d^s : X \times X \to [0, \infty)$ defined by

$$d^s(x,y) = \begin{cases} 0 & \text{if} \quad x = y; \\ (\frac{1}{4})^{\frac{1}{3}} & \text{if} \quad x \neq y \in \{0,\, 1\}; \\ |x-y|^{\frac{1}{3}} & \text{if} \quad x \neq y \in \{0\} \cup \{\frac{1}{2n},\, \geq 1\}; \\ (\frac{1}{4})^{\frac{1}{3}} & \text{Otherwise.} \end{cases}$$

On symmetric completions of a quasi b-metric space

Definition Salbany, 1974 Let (X, p) be a quasi metric space. We say that a quasi metric space $(\bar{X}, \bar{p})$ is a **bicompletion** of (X, p) if

- (i) $(\bar{X}, \bar{p})$ is bicomplete;
- (ii) $X \subseteq \bar{X}$, and $\bar{p}|_{X \times X} = p$;
- (iii) there exists $T : (X, p) \to (\bar{X}, \bar{p})$, such that T is an isometry, that is $\bar{p}(Tx, Ty) = p(x, y)$ for all $x, y \in X$ and TX is $\bar{p}^s$-dense in $\bar{X}$.

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Salbany, 1974 Theorem

Let (X, p) be a quasi metric space. Then:

- The space (X, p) has a bicompletion $(\bar{X}, \bar{p})$.
- The bicompletion of (X, p) is unique in the sense that if $(\bar{X}, \bar{p})$ and $(\tilde{X}, \tilde{p})$ are the two bicompletions of (X, p), then there exists a bijective isometry $T : \bar{X} \to \tilde{X}$, which restricts to the identity on X.

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The following can be regarded in some sense as a quasi metric space extension of a quasi b-metric space.

Proposition

Let (X, α, p) be a quasi b-metric space. Then there exists a bicomplete quasi metric space $(\bar{X}, \bar{d})$ such that,

- $X \subseteq \bar{X}$,
- Every Cauchy sequence in (X, α, p) converges to a point in $(\bar{X}, \bar{d})$ with respect to $\bar{d}^s$.

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The following can be regarded in some sense as a metric completion of a quasi b-metric space:

Proposition

Let (X, α, p) be a quasi b-metric space. Then there exists a complete metric space $(\bar{X}, \bar{d}^s)$ such that,

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For symmetric structures:

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Let (X, α, p) be a b-metric space. Then there exists a complete metric space $(\bar{X}, \bar{d})$ such that,

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Definition

Let (X, α, p) be a quasi b-metric space. We say that a quasi b-metric space $(\bar{X}, \beta, \bar{p})$ is a **bicompletion** of (X, α, p) if

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Theorem

Every quasi b-metric space (X, α, p) admits a quasi b-metric bicompletion $(\bar{X}, \beta, \bar{p})$, with $\beta = 4\alpha^3$.

Since the bicompletion of a quasi-metric space is unique up isometry:

Theorem

The quasi b-metric bicompletion of a quasi b-metric space is unique up to isometry.

Nguyen van Dung et al, 2018 Theorem

Every b-metric space (X, α, p) admits a b-metric completion $(\bar{X}, \beta, \bar{p})$.

Nguyen van Dung et al. 2018 Theorem

A completion of a b-metric space is unique up to isometry.

On characterizations of bicompleteness for quasi b-metric spaces

Let M_{qbms} denote the class of all quasi b-metric spaces. For $(\bar{X}, \beta, \bar{p})$ and (X, α, p) in M_{qbms}. We say that $(\bar{X}, \beta, \bar{p})$ contains (X, α, p) when $X \subseteq \bar{X}$, and $\bar{p}|_{X \times X} = p$, with $\alpha \leq \beta$. We say that (X, α, p) is M_{qbms}-convergence complete, if for a sequence $\{x_n\}$ in X that $\bar{p}^s$-converges to a point $\bar{x} \in \bar{X}$, there exists $x \in X$, such that $\{x_n\}$ converges to x with respect to p^s, for every $(\bar{X}, \beta, \bar{p})$ that contains (X, α, p).

Theorem

Let $(X, \alpha, p) \in M_{qbms}$, then the following are equivalent.

- The space (X, α, p) is bicomplete;
- The space (X, α, p) is M_{qbms}-convergence complete.

Theorem

Let $(X, \alpha, p) \in M_{qbms}$, then the following are equivalent.

- The space (X, α, p) is bicomplete;
- The space (X, α, p) is M_{qbms}-convergence complete.

Corollary

Let (X, α, p) be a b-metric space and M_{qbms} be restricted to the class of b-metric spaces. Then the following are equivalent:

- The space (X, α, p) is complete;
- The space (X, α, p) is M_{qbms}-convergence complete.

Corollary

Let (X, α, p) be a b-metric space and M_{qbms} be restricted to the class of b-metric spaces. Then the following are equivalent:

- The space (X, α, p) is complete;
- The space (X, α, p) is M_{qbms}-convergence complete.

In Conclusion:

- We presented basic properties of quasi b-metric spaces as an asymmetric structure for b metric spaces.
- We constructed explicity a quasi-metric (metric) d on X from a quasi b-metric space (X, α, p), and discussed the bicompletions of quasi b-metric spaces (or their extensions) and shown that such bicompletions are unique up to isometry.

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Thank you

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