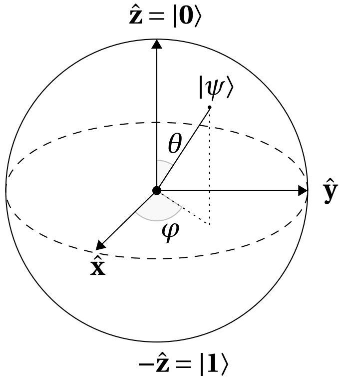
A Topological Bottlekneck in Quantum Information

by Stuart Wayland

A few disclaimers...

The Space of Quantum States (2 dimensions)



The Bloch Sphere

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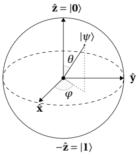

- A 2 dimensional quantum state ("qubit")

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{C}^2$$

- Some helpful structure...

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \sim \begin{pmatrix} \lambda w_1 \\ \lambda w_2 \end{pmatrix}, \qquad \lambda = re^{i\theta} \in \mathbb{C}^*$$

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- g establishes a convenient isomorphism to $\mathbb{S}^2$

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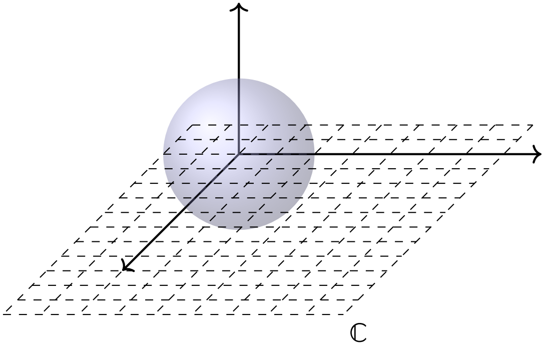

SUMTOPO 2024

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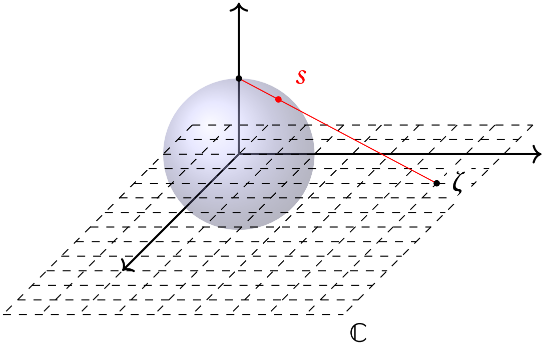

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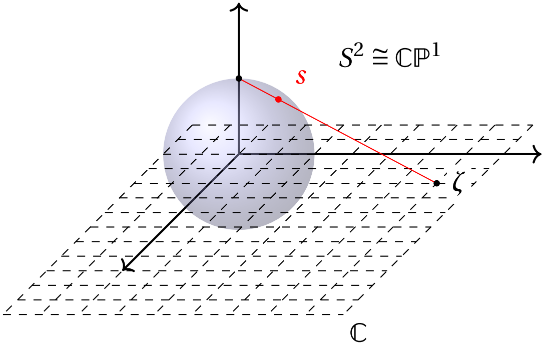

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- $H = g \circ f : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf Fibration

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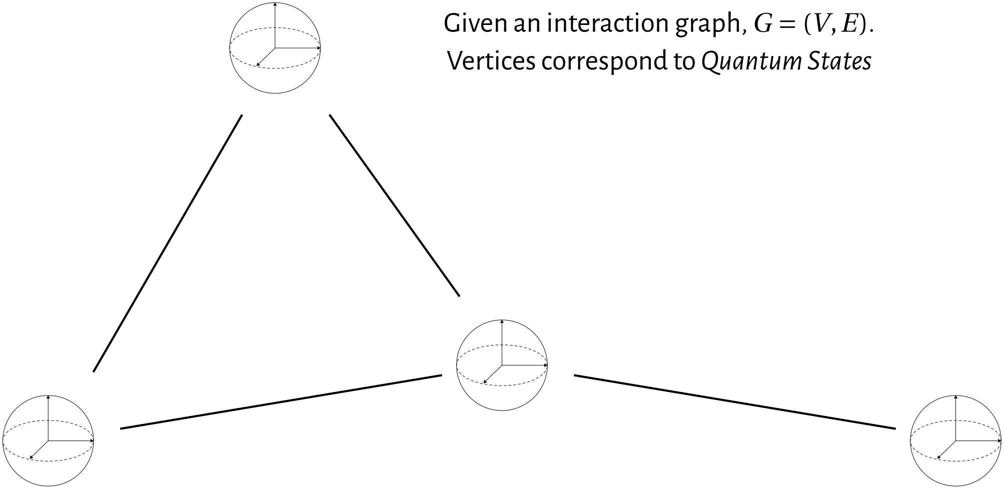
- The orbit of $SU(2)$ on $w \in \mathbb{CP}^1$ generates our space

$$SU(2)/U(1) \cong \mathbb{S}^3/\mathbb{S}^1 \cong \mathbb{CP}^1$$

Why is this representation useful?

Spin Glass Systems



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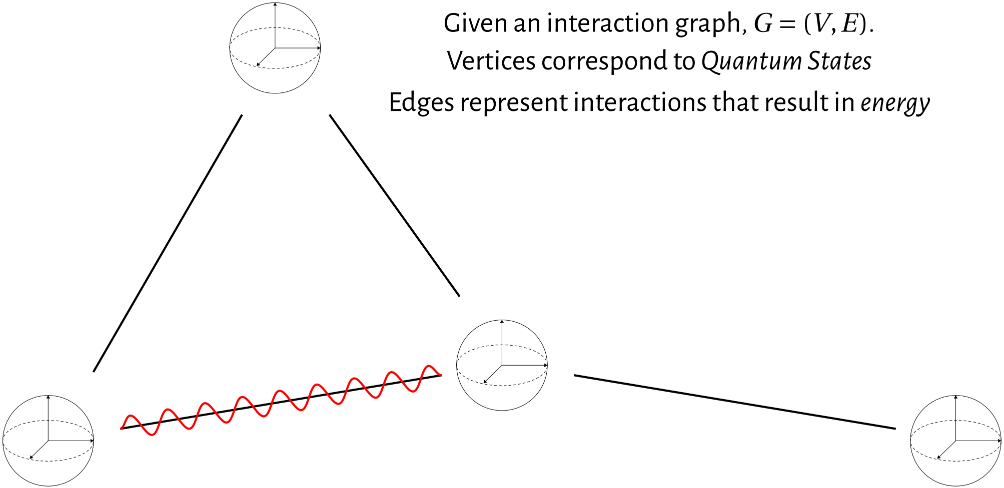

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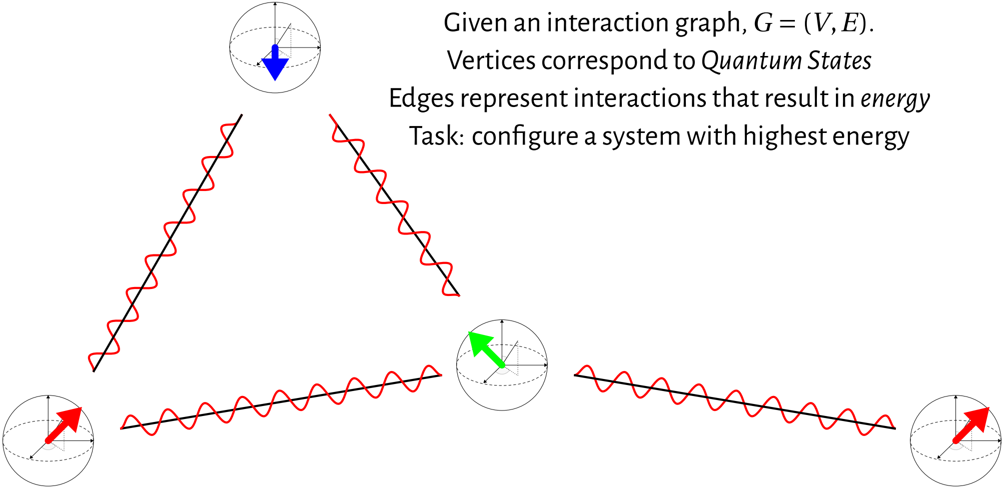

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Task: configure a system with highest energy

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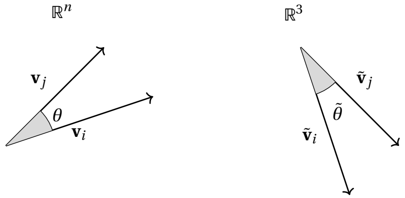
- Getting back to the space of Quantum States is called **Rounding**
  - we need to preserve inner products:

$$v_i \cdot v_j \approx \tilde{v}_i \cdot \tilde{v}_j$$

Relaxation and Rounding



- Hyperplane rounding (getting to S^2)

$$\mathbf{Z} \sim \mathcal{N}(0,1)^{3\times n}$$

- Then, we may define

$$\tilde{\mathbf{v}}_i := \frac{Z\mathbf{v}_i}{||Z\mathbf{v}_i||}$$

- This lands $\tilde{v}_i$ right on S^2, **and** $\theta \approx \tilde{\theta}$. We are back to the land of quantum states!

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3. interesting complexity implications (take my word for it)

So what am I doing at a topology conference?

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$$\begin{tikzcd}
\mathbb{C}^d \arrow[d] \arrow[dr] & \\
\mathbb{S}^{2d-1} \arrow[r] & \mathbb{CP}^{d-1} \cong \mathbb{S}^{2d-1}/\mathbb{S}^1 \not\cong \mathbb{S}^{2d}
\end{tikzcd}$$

The space of Quantum states - $\mathbb{CP}^{d-1}$

What we know

- homeomorphic to $\mathbb{S}^{2d-1}/\mathbb{S}^1$
- has real dimension $2d-2$ and is a submanifold of $\mathbb{R}^{d^2-1}$
- is a compact and connected Hausdorff space
- is generated by $SU(d)$

What we need

- A *good* representation of $\mathbb{CP}^{d-1}$

Thank you!