On Cauty's example of a metric linear space without the extension property

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Cauty's Example

(Cauty 1995)

There exists a metric linear space C which is not an absolute extensor for metric spaces.

Moreover, C is sigma-compact.

Definition

A space X is an absolute extensor for metric spaces if every mapping

$$f : A \to X,$$

where A is a closed subset of a metric space Z, extends to Z.

If every such f extends to a neighborhood of A in Z, X is called an absolute neighborhood extensor.

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Fact

Every locally convex topological vector space is an absolute extensor.

Observation

Cauty's Example shows that the local convexity cannot be dropped.

Remark

For sigma-compact metric linear space E, the following are equivalent:

(1) E is an absolute extensor for metric spaces,

(2) E is an absolute extensor for (metric) compacta.

(3) For every compactum $A \subset E$, the identity map $\mathrm{id}_A : A \to E$ can be approximated by maps

$$\phi : A \to E$$

so that $\phi(A)$ is finite-dimensional.

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Placing Cauty's space C in $C^*(K)$

- Identify the compactum K with

$$\{\delta_k : k \in K\}$$

the set of Dirac measures in $C^*(K)$ with the weak* topology.

- For every n, let

$$L_n(K) = \{\sum_{i=1}^{n} t_i\delta_{k_i} : \; |t_1| + \cdot + |t_n| \leq n, \; k_1, \ldots, k_n \in K\}$$

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- Then

$$L(K) = \bigcup_{n=1}^{\infty} L_n(K) = \text{span}(K)$$

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The finest vector topology τ_0 on $L(K)$

• U is open in $(L(K), \tau_0)$ iff, for every n,

$$U \cap L_n(K) \quad \text{is open in} \quad L_n(K).$$

• By [Turpin, 1976], the sets

$$\overline{U}_0 \cap \bigcap \overline{L_n(K) + U_n},$$

where U_n, $n \geq 0$, are weak* open neighborhoods of 0 in $C^*(K)$ and $\overline{U_n}$ their weak$^*-$closures, form a base of neighborhoods of 0 in $(L(K), \tau_0)$.

• For any metric topology τ on $LK)$, there exists a finer metric topology τ' so that the completion of $(L(K), \tau')$ has FDD ; in particular, has a sequence of continuous functionals separating points.

Here: FDD is f-d decomposition property, that is, for every $x \in L(K)$

$$x = \sum_{n=1}^{\infty} T_n(x),$$

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Fact

(1) Since vector operations are τ_0- continuous

$$(L(K), \tau_0) \text{ is a t.v.s.}$$

(2) the original and τ_0- topologies coincide on K and each $L_n(K)$; thus,

- K is a Hamel basis for $L(K)$;
- $L(K)$ is sigma-compact.

(3) $L(K)$ is a free t.v.s. over K and (for nontrivial K) nonlocally convex.

(4) For any continuous mapping $f : K \to F$, where F is a t.v.s., there is unique continuous linear operator

$$F : (L(K), \tau_0) \to F \text{ such that } F|K = f;$$

in particular, such an operator $F : (L(M), \tau_0) \to (L(K), \tau_0)$ exists for a continuous surjection $f : M \to K$. Furthermore, F is open.

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Cauty' Example - a precise version

(Cauty 1995)

There exists a compactum K and a metric linear topology τ on $L(K)$ such that:
For any metric linear topology τ' on $L(K)$, $\tau_0 \subset \tau' \subset \tau$,

$$(L(K), \tau') \quad \text{is not an absolute extensor for compacta.}$$

Additionally, the map $\mathrm{id}_K : K \to (L(K)), \tau')$ cannot be approximated by maps

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The compactum K

• [Dranishnikov, 1988]

$M =: S^7$ has a partition $\mathcal{P}$ into CE compacta so that the quotient space

$$M/\mathcal{P} =: K \quad \text{is an infinite-dimensional compactum.}$$

• [Walsh, 1976]

The partition $\mathcal{P}$ can be further enhanced so that the quotient map

$$f : M \to K \quad \text{is an open mapping.}$$

• Each pre-image $f^{-1}(x)$, $x \in K$, is a CE compactum, meaning

$$f^{-1}(x) = \bigcap B_k, \; B_{k+1} \subset B_k, \; B_k \text{ is a copy of Euclidean ball.}$$

Remark

The topological sine curve is a nontrivial CE compactum.

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The operator $F : L(M) \to L(K)$

• For $M = S^7$, consider $\mathrm{span}(M) \subset C^*(M)$ and let

$$L(M) =: (\mathrm{span}(M), \tau_0),$$

where τ_0 is the finest vector topology.

• Since M and K are Hamel basis of $L(M)$ and $L(X)$, $f : M \to K$ there exists a unique continuous open operator

$$F : L(M) \to L(X) \text{ such that } F|M = f.$$

• If, for some metrics d', d, and an open set $U \subset (L(K), d)$,

(i) $F : (F^{-1}(U), d') \to (U, d)$ is continuous and

(ii) $(L(K), d)$ is an absolute extensor

then homotopy types of $F^{-1}(U)$ and U are the same.

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Similarities and differences between $L(M)$ and $L(K)$

• Both $L(M)$ and $L(K)$ are sigma-compact.

• While $L(K)$ is not, $L(M)$ is a countable union of f-d compacta.
Conclusion: $L(M)$ is an absolute extensor in any metric linear topology.

• For each n, define $H_n(K) \subset L(K)$ (and similarly $H_n(M) \subset L(M)$) by

$$H_n(K) = \Big\{ \sum_{i=1}^{n} t_i x_i : \ x_i \in K, \{x_i\} \text{ distinct, and } t_i \neq 0 \Big\}.$$

• We have $\bigcup_{n=1}^{\infty} H_n(K) = L(K) \setminus \{0\}$ and $\bigcup_{n=1}^{\infty} H_n(M) = L(M) \setminus \{0\}$.

• Both $H_n(M)$ and $H_n(K)$ are metric local compacta (in the free topologies).

• While each $H_n(K)$ is not, $H_n(M)$ is an absolute **neighborhood** extensor.

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Key facts validating the choice of K

• Write 2^M for the hyperspace of all compacta in M with the Hausdorff topology. The set-valued function

$$K \to 2^M \quad \text{given by } x \to f^{-1}(x)$$

has CE images and, due to openness of f, is continuous.

• More generally, the set-valued function $\phi_n : H_n(X) \to 2^{H_n(M)}$ defined by

$$\phi_n(t_1x_1 + \cdots + t_nx_n) = t_1f^{-1}(x_1) + \cdots + t_nf^{-1}(x_n)$$

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Remark

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$$t_1 f^{-1}(x_1) + \cdots + t_n f^{-1}(x_n)$$

is homeomorphic to $f^{-1}(x_1) \times \cdots \times f^{-1}(x_n)$, which is CE because each $f^{-1}(x_i)$ is.

Near-selection Theorem, [Haver, 1978]

Let

(i) Z be a metric space which is a countable union of f-d compacta,

(ii) T be a metric absolute neighborhood extensor.

(iii) $\psi : Z \to 2^T$ be a set-valued mapping with CE images, and

(iv) $\epsilon : Z \to (0, 1]$ be a continuous function.

Then there exists $\chi : Z \to T$ such that

$$\text{dist}(\chi(z)), \psi(z)) < \epsilon(z), z \in Z;$$

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Near-selection Theorem at work

Corollary

We have

- $Z =: F^{-1}(H_n(K)) \subset L(M)$ is countable union of f-d compacta;
- $T =: H_n(M)$ is an absolute neighborhood retract
- $\psi =: \phi_n \circ F : F^{-1}(H_n(K)) \to 2^{H_n(M)}$ is continuous and has CE images.

By Near-selection Theorem, for every continuous function $\epsilon : F^{-1}(H_n(K)) \to (0, 1]$ there exists

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Infinite-dimensionality of the compactum K in use

• A classic test for infinite-dimensionaliy:

For some closed set $A \subset K[\subset L(K)]$ some mapping

$$g : A \to S^8$$

has no extension over K.

• However, g can be extended to

$$\bar{g} : W \to S^8,$$

where W is a closed neighborhood of A in $L(K)$.

• Since $8 > 7$,

$$\bar{g} \circ [f|f^{-1}(W)] : f^{-1}(W) \to S^8$$

extends to $h_0 : M = S^7 \to S^8$.

Additionally, since $M \cap F^{-1}(W) = f^{-1}(W)$, h_0 extends over $F^{-1}(W)$ to

$h_0 : M \cup F^{-1}(W) \to S^8$ and satisfies $h_0|F^{-1}(W) = \bar{g} \circ [F|F^{-1}(W)]$.

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Goal: Extend h_0 to h on $F^{-1}(U \cup W)$, $K \subset U \subset L(K)$

- An open $U \subset L(K)$ ($K \subset U$) and $h : F^{-1}(U) \to S^8$ are inductively constructed such that

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$$G_n(X) = \{z \in L(X) : z = t_1 x_1 + \cdots t_n x_n, x_i \in X, -\infty < t_i < \infty\}$$

$$H_n(X) = G_n(X) \setminus G_{n-1}(X); \quad \text{here, } G_0(X) = \{0\}.$$

- Inductively, the open sets $U_n = U \cap G_n(K)$ are constructed. Finally,

$$U = \bigcup (U \cap G_n(K)).$$

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Using U, W, and h to find τ on $L(K)$ claimed by Cauty

- There are metric vector topologies τ on $L(K)$ and τ' on $L(M)$ (use Birkhoff–Kakutani technique) such that

(i) U, W are $\tau-$open and $\bar{g} : (W, \tau) \to S^8$ is continuous, and

(ii) $F : (L(M), \tau') \to (L(K), \tau)$ and $h : (F^{-1}(U), \tau') \to S^8$ are continuous.

- **Assume $(\mathbf{L}(\mathbf{K}, \tau))$ is an absolute extensor.**

Then, (U, τ) is an absolute neighborhood extensor. Then, by linearity of F, there exists a fine homotopy inverse

$$\psi : (U, \tau) \to (F^{-1}(U), \tau') \text{ of}$$

$$F : (F^{-1}(U), \tau') \to (U, \tau), \text{ that is ,}$$

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g extends over K if for some $\alpha : K \to S^8$ and $H : A \times [0,1] \to S^8$ we have

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• If $\psi : U \to F^{-1}(U)$ is a homotopy inverse of $F : F^{-1}(U) \to U$, then

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(Construction of h_1)

- Recall: $h_0 : M \cup F^{-1}(W) \to S^8$.

Extend h_0 over open P_0 in $L(M)$, $M \cup F^{-1}(W) \subset P_0$, and call it also h_0.

- Define

$$M_1 = \{z \in H_1(K) : \phi_1(z) \subset P_0\},$$

that is, for $z = tx, x \in K, t \neq 0$, $\phi_1(tx) = tf^{-1}(x)$ must be $\subset P_0$.

- $K \subset M_1$ and $M_1 \subset H_1(K)$ is open by continuity of

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• $K \subset M_1$ and $M_1 \subset H_1(K)$ is open by continuity of

$$\phi_1 : H_1(K) \to 2^{H_1(M)}.$$

• Find open $U_1, U \subset H_1(X)$ with

$$K \cup (W \cap H_1(K)) \subset U_1 \subset \overline{U}_1 \subset U \subset \overline{U} \subset M_1.$$

• By Near-selection Thm applied to $\psi_1 =: \phi_1 \circ [\mathbf{F}|\mathbf{F}^{-1}(\mathbf{U} \setminus \mathbf{W})]$, get

$$\chi_1 : F^{-1}(U \setminus W) \to H_1(M) \text{ with } d_1(\chi_1(y), \psi_1(y) < d_1(y, F^{-1}(W)).$$

• **Remark**: Above use a continuous metric d_1 on $L(M)$ such that

$F^{-1}(W)$ is d_1 – closed and $h_0 : F^{-1}(W) \to S^8$ is d_1 – continuous.

• Write $[y_1, y_2] = \{ty_1 + (1-t)y_2 : 0 \leq t \leq 1\}$ and let

$$R_1 = \{y \in F^{-1}(M_1 \setminus W) : [y, \chi_1(y)] \subset P_0.\},$$

$R_1 \subset F^{-1}(H_1(K) \cup W)$ is open.

• Pick open $V \subset F^{-1}(H_1(K) \cup W)$ with

$$F^{-1}(W) \subset V \subset \overline{V} \subset R_1.$$

• Let $\lambda : F^{-1}(H_1(K) \cup W \to [0,1]$ such that

$$\lambda(\overline{V}) = 0 \text{ and } \lambda(y) = 1 \text{ if} y \notin R_1.$$

• Define, for $y \notin V$, $h_1(y) = h_0\big((1-\lambda(y))y + \lambda(y)\chi_1(y)\big)$; clearly $h_1|\overline{V} = h_0$.

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