Canonical extensions of frames via fitted sublocales and strongly exact filters

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Polarities à la Birkhoff

A **polarity** is $P = (X, Y, Z)$ with $Z \subseteq X \times Y$. It induces

$$
\begin{tikzcd}
\wp(X) \arrow[r, bend left, "p"] & \wp(Y) \arrow[l, bend left, "q"]
\end{tikzcd}
\quad \text{with} \quad
\begin{aligned}
p(M) &= \{y \mid \forall x \in M.\ xZy\} \\
q(N) &= \{x \mid \forall y \in N.\ xZy\}
\end{aligned}
$$

The complete lattice of Galois closed sets

$$\mathcal{G}(P) = \mathrm{fix}(q \circ p) = \{M \in \wp(X) \mid q(p(M)) = M\}$$

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The complete lattice of Galois closed sets

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Theorem (folklore)

The induced mappings $X \xrightarrow{i_X} \mathcal{G}(P) \xleftarrow{i_Y} Y$ *unique such that*

- $\mathrm{im}(i_X)$ $\bigvee$*-generates and* $\mathrm{im}(i_Y)$ $\bigwedge$*-generates* $\mathcal{G}(P)$
- $i_X(x) \leq i_Y(y)$ *iff* xZy

Canonical extensions of lattices

For a bounded distributive lattice D,

$$D^\delta = \mathcal{G}(\mathrm{Filt}(D), \mathrm{Idl}(D), Z) \quad \text{where} \quad FZI \iff F \cap I \neq \emptyset$$

Theorem

For the Priestley dual X of D, $D^\delta \cong \mathrm{Up}(X)$.

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Further facts

- D^δ is a frame & coframe.
- The embedding $e : D \to \mathsf{Idl}(D) \to D^\delta$ uniquely identifies D^δ, by properties: *compactness* and *density*
- Duality between X and D recovered algebraically from e.
- Extensions of (monotone) maps $D \to E$ to $D^\delta \to E^\delta$
- Preservations of equations, e.g. if D is Boolean then so is D^δ.
- ...

Frame theory in 2 minutes

A **frame** L is a complete lattice such that, for any $A \subseteq L, b \in L$

$$(\bigvee A) \wedge b = \bigvee_{a \in A} (a \wedge b)$$

Consequently: $a \wedge b \leq c$ iff $a \leq b \to c$

Example: for a space X, its lattice of opens $\Omega(X)$ is a frame.

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Example: for a space X, its lattice of opens $\Omega(X)$ is a frame.

Conversely, every frame L gives a space $\Sigma(L) = (\mathrm{CP}(L), \tau_L)$ where

- $\mathrm{CP}(L) = \{$ filters $P \subseteq L \mid \bigvee A \in P$ implies $A \cap P \neq \emptyset\}$
- τ_L generated by $\{P \in \mathrm{CP}(L) \mid a \in P\}$ for every $a \in L$

$$\begin{tikzcd} \mathrm{Top} \arrow[r, bend left, "\Omega"] & \mathrm{Frm} \arrow[l, bend left, "\Sigma"] \end{tikzcd}$$

This identifies spacial frames and sober spaces.

well... another 2 minutes: Sublocales

For every frame L,

$$\mathcal{S}(L)$$

is the coframe of **sublocales**, i.e. subsets

$$S \subseteq L$$

such that

- S is closed under $\bigwedge$ and
- $x \in L$ and $s \in S$ implies $x \to s \in S$.

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For $a \in L$, special sublocales:

- **open** sublocale $\mathfrak{o}(a) = \{x \in L \mid a \to x = x\}$ (representing a)
- **closed** sublocale $\mathfrak{c}(a) = {\uparrow}a$ (its complement)

Canonical extensions of frames

For frame $L = \mathsf{Idl}(D)$,

$$\mathsf{Filt}(D) \cong \mathsf{SO}(L) \quad \text{(i.e. \textit{Scott-open} filters on } L\text{).}$$

Therefore,

$$D^\delta \;\cong\; L^{\mathsf{SO}} := \mathcal{G}(\mathsf{SO}(L), L, Z) \quad \text{where} \quad FZa \iff F \ni a$$

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Works for any frame L:

- The induced mapping $e : L \to L^{\mathrm{SO}}$ is a pre-frame homo.
- It is frame embedding iff L pre-spacial.
- L stably locally compact $\Rightarrow L^{\mathrm{SO}}$ frame & coframe (constructively)
- $L = \mathrm{Idl}(B) \Rightarrow L^{\mathrm{SO}} \cong \mathcal{S}_c(L)$
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Question: What if we replace Scott-open filters?

Filters on frames

The geometric picture behind $\mathcal{G}(\mathsf{SO}(L), L, \ni)$:

$$\begin{tikzcd} \mathrm{Filt}(L) \arrow[r, bend left, "\mathrm{su}"] & \mathcal{S}(L) \arrow[l, bend left, "\mathrm{fi}"] \end{tikzcd} \qquad \text{where} \qquad \begin{aligned} \mathrm{su}(F) &= \bigcap_{a \in F} \mathfrak{o}(a) \\ \mathrm{fi}(S) &= \{a \mid S \subseteq \mathfrak{o}(a)\} \end{aligned}$$

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Our intuition: Filters approximate sublocales $\Rightarrow$ order them by

$$F \sqsubseteq G \quad \text{iff} \quad F \supseteq G \quad \Rightarrow \quad \underline{\mathsf{Filt}(L) \text{ is a coframe!}}$$

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Monotone adjunction:

$$\mathrm{fi}(S) \sqsubseteq F \iff S \subseteq \mathrm{su}(F)$$

Classes of filters

Fixpoints of $\mathrm{fi} \dashv \mathrm{su}$, i.e. filters $F \subseteq L$ such that

$$\mathrm{su}(F) \subseteq \mathfrak{o}(a) \iff F \ni a$$

are precisely the **strongly exact** filters $\mathrm{SE}(L) \subseteq \mathrm{Filt}(L)$.

[Moshier–Pultr–Suarez'20]

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$$\begin{tikzcd}
\mathfrak{B}(\mathrm{Filt}(L)) \arrow[r, hook] & \mathrm{E}(L) \arrow[dr, hook] & & \\
& & \mathrm{SE}(L) \arrow[r, hook] & \mathrm{Filt}(L) \\
\mathcal{J}(\mathrm{CP}(L)) \arrow[r, hook] & \mathcal{J}(\mathrm{SO}(L)) \arrow[ur, hook, "\text{[Johnstone'85]}"'] & &
\end{tikzcd}$$

Here $\mathfrak{B}(-)$ is the co-Booleanization operation,
$\mathrm{E}(L)$ = exact filters, $\mathrm{CP}(L)$ = completely prime filters.

Filter extensions

Given a class $\mathcal{F} \subseteq \mathsf{Filt}(L)$ define

$$L^{\mathcal{F}} = \mathcal{G}(\mathcal{F}, L, \ni) \qquad \text{(order denoted } \sqsubseteq\text{)}$$

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Theory of polarities $\Rightarrow$ the induced maps

$$\mathcal{F} \xrightarrow{\kappa} L^{\mathcal{F}} \xleftarrow{e} L$$

uniquely determined by:

(density) $\mathrm{im}(\kappa)$ $\bigsqcup$-generates and $\mathrm{im}(e)$ $\bigsqcap$-generates $L^{\mathcal{F}}$

(compact.) $\kappa(F) \sqsubseteq e(a)$ iff $F \ni a$

e is enough as $\kappa(F) = \bigsqcap_{a \in F} e(a)$.

Properties of filter extensions

Basic facts

- $\kappa\colon \mathcal{F} \to L^{\mathcal{F}}$ order embedding
- $e\colon L \to L^{\mathcal{F}}$ preserves $0, 1, \wedge$
- e order embedding iff e injective iff L is $\mathcal{F}$-**separable**:

$$a = b \quad \text{iff} \quad \forall F \in \mathcal{F} \quad a \in F \iff b \in F$$

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Observation: $L^{\mathcal{F}} \;\cong\; \mathcal{J}(\mathcal{F}) = \{\bigsqcup M \mid M \subseteq \mathcal{F}\}$

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Theorem

$L^{\mathcal{F}} \subseteq \mathrm{Filt}(L)$ *is a subcolocale inclusion if,* $\forall F \in \mathcal{F}\ \forall a \in L$,

$$F \searrow \uparrow a = \{x \mid x \vee a \in F\} \;\in\; \mathcal{F}$$

Classes of sublocales corresponding to filters

Subcolocale inclusions (by the previous theorem)

$$\begin{tikzcd}
\mathsf{SE}(L) \arrow[r, hook] \arrow[d, leftrightarrow, "\cong"] & \mathsf{Filt}(L) \arrow[d, bend left, "\mathsf{su}"] \\
\mathcal{S}_o(L) \arrow[r, hook, dashed] & \mathcal{S}(L) \arrow[u, bend left, "\mathsf{fi}"]
\end{tikzcd}$$

$\mathcal{S}_o(L) = \{\text{intersections of opens}\}$

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\mathsf{E}(L) \arrow[dr, hook] & & \\
& \mathsf{SE}(L) \arrow[r, hook] \arrow[dd, leftrightarrow, "\cong"] & \mathsf{Filt}(L) \arrow[dd, bend left, "\mathsf{su}"] \\
\mathtt{fit}[\mathcal{S}_b(L)] \arrow[dr, hook] & & \\
& \mathcal{S}_o(L) \arrow[r, hook, dashed] & \mathcal{S}(L) \arrow[uu, bend left, "\mathsf{fi}"]
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$$

exact filters: $\bigwedge M$-closed for
$(\bigwedge M) \vee b = \bigwedge_{a \in M}(a \vee b)$

$$\mathcal{S}_b(L) = \mathcal{J}(\{\text{complemented sublocales}\})$$
$$= \mathcal{J}(\{\mathfrak{c}(y) \cap \mathfrak{o}(x) \mid x, y \in L\})$$

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\end{tikzcd}$$

regular filters: $\{1\} \diagdown F$

$\mathcal{S}_c(L) = \mathcal{J}(\{\text{closed sublocales}\})$

Classes of sublocales corresponding to filters

Subcolocale inclusions (by the previous theorem)

$$\mathcal{S}_k(L) = \mathcal{J}(\{\text{compact sublocales}\})$$

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 & \mathsf{SE}(L) \arrow[r, hook] \arrow[dd, leftrightarrow, "\cong"] & \mathsf{Filt}(L) \arrow[dd, bend left, "\mathsf{su}"] \\
\mathcal{J}(\mathsf{SO}(L)) \arrow[ur, hook] & & \\
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 & & \mathcal{S}_o(L) \arrow[r, hook, dashed] & \mathcal{S}(L) \arrow[uu, bend left, "\mathsf{fi}"] \\
\mathtt{fit}[\mathcal{S}_{sp}(L)] \arrow[r, hook] & \mathtt{fit}[\mathcal{S}_k(L)] \arrow[ur, hook] & &
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$\mathcal{S}_{sp}(L) = \{\text{spatial sublocales}\}$

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Remarks

The proof of $\mathfrak{B}(\mathsf{Filt}(L)) \cong \texttt{fit}[\mathcal{S}_c(L)]$ uses

- $\mathfrak{B}(\mathsf{Filt}(L)) = \mathcal{J}(\mathsf{C}(L))$ where $\mathsf{C}(L) = \{\mathsf{fi}(\mathfrak{c}(a)) \mid a \in L\}$

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- $\mathsf{E}(L) = \mathcal{J}(\mathsf{LC}(L))$ where

  $\mathsf{LC}(L) = \{\mathsf{fi}(\mathfrak{o}(y) \smallsetminus \mathfrak{o}(x)) \mid x, y \in L\} = \{{\uparrow}y \smallsetminus {\uparrow}x \mid x, y \in L\}$

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$\Rightarrow$

$$L^{\mathsf{C}} \cong \mathcal{G}(\mathfrak{c}[L], \mathfrak{o}[L], \subseteq) \cong \mathtt{fit}[\mathcal{S}_c(L)] \cong \mathsf{int}_{\mathfrak{c}[L]}[\mathcal{S}_o(L)]$$
$$L^{\mathsf{CP}} \cong \mathcal{G}(\mathcal{S}_*(L), \mathfrak{o}[L], \subseteq) \cong \mathtt{fit}[\mathcal{S}_{sp}(L)] \cong \mathsf{sp}[\mathcal{S}_o(L)]$$

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$\mathsf{Filt}(L)$ as a geometric extension of L

Define

- $\mathfrak{of}(a) = \mathsf{fi}(\mathfrak{o}(a)) = {\uparrow}a$
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$F \in \mathsf{Filt}(L)$ exact iff $\exists \{x_i, y_i\}_i$ s.t. $F = \bigsqcup_i (\mathfrak{of}(y_i) \smallsetminus \mathfrak{of}(x_i))$

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L subfit $\Rightarrow$ $\mathsf{E}(L)$ Boolean since $\mathfrak{B}(\mathsf{Filt}(L)) = \mathcal{J}(\mathsf{C}(L)) \subseteq \mathsf{E}(L)$ and

$$\mathfrak{of}(y) \smallsetminus \mathfrak{of}(x) \overset{(sfit)}{=} \bigsqcup_{\mathfrak{cf}(b) \sqsubseteq \mathfrak{of}(y)} (\mathfrak{cf}(b) \smallsetminus \mathfrak{of}(x)) = \bigsqcup_{\mathfrak{cf}(b) \sqsubseteq \mathfrak{of}(y)} \mathfrak{cf}(b \vee x)$$

Closing words

- New characterisations of subfitness
- New tools for working with $\mathcal{S}_c(L)$ resp. $\mathsf{E}(L)$ – the latter described as joins of locally closed filters, universal properties given by polarities.
- $\mathsf{E}(L)$ contains closed, open, locally closed filters $\rightarrow$ justifies why it is a suitable (geometric) discretization of L?

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Open problems

- When is $L^{\mathcal{F}}$ a frame? (e.g. for $\mathcal{F} = \mathsf{E}(L)$)
- Is there a natural simple class $\mathcal{F}$ such that $\mathsf{SE}(L) = \mathcal{J}(\mathcal{F})$?
- Theory of extensions of maps $L \to M$ to $L^{\mathcal{F}} \to M^{\mathcal{F}}$.
- Preservation of topological properties?
  (e.g. L subfit $\Rightarrow$ L^{E} Boolean, when 0-dimensional like $\mathcal{S}(L)$?)

Thank you!