# Fans of Cook continua

Teja Kac

Joint work with Iztok Banič and Matevž Črepnjak

University of Maribor

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**Definition**

An inverse sequence is a double sequence $\{X_n, f_n\}_{n=1}^{\infty}$ of metric spaces $X_n$ and functions $f_n : X_{n+1} \to X_n$. The spaces $X_n$ are called the coordinate spaces and the functions $f_n$ are called the bonding functions. The inverse limit of an inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ is the subspace of the product space $\prod_{n=1}^{\infty} X_n$ that consists of all points $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{n=1}^{\infty} X_n$ such that $x_n = f_n(x_{n+1})$ for each positive integer $n$:

$$\varprojlim\{X_n, f_n\}_{n=1}^{\infty} = \left\{(x_1, x_2, x_3, \ldots) \in \prod_{n=1}^{\infty} X_n \mid x_n = f_n(x_{n+1}) \text{ for each } n\right\}.$$

## Definition

Let $n$ be a positive integer. A nondegenerate continuum $X$ is $\frac{1}{n}$-rigid, if

$$f^n = 1_X$$

for any continuous surjective function $f : X \to X$. If a continuum $X$ is not $\frac{1}{n}$-rigid for any positive integer $n$, then we say that $X$ is 0-rigid. If a continuum $X$ is $\frac{1}{1}$-rigid, then we also say that $X$ is 1-rigid or rigid.

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Let $X$ be a continuum. We define the degree of rigidity, $\mathrm{degrig}(X)$, of the continuum $X$ as follows:

$$\mathrm{degrig}(X) = \min\left(\{n \in \mathbb{N} \mid X \text{ is } (1/n)\text{-rigid}\} \cup \{\infty\}\right).$$

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# Cook continuum

### Definition

A nondegenerate continuum $X$ is a Cook continuum if for any nondegenerate subcontinuum $S$ of $X$ and any non-constant continuous function $f : S \to X$, it follows that

$$f(x) = x$$

for any $x \in S$.

# Properties of Cook continua

- Let $X$ be a Cook continuum and let $x_0 \in X$ be any point. Then $x_0$ is a non-cut point for $X$.
- Let $X$ be a Cook continuum and let $S$ be a nondegenerate subcontinuum of $X$. Then $S$ is a Cook continuum.
- Let $X$ and $Y$ be any Cook continua, let $S$ be a nondegenerate subcontinuum of $X$, and let $\varphi : X \to Y$ be a homeomorphism. Then

$$f = \varphi|_S$$

for any non-constant continuous function $f : S \to Y$.
- Any Cook continuum $X$ is rigid (1-rigid) and $\text{degrig}(X) = 1$.

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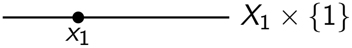
# Stars of continua



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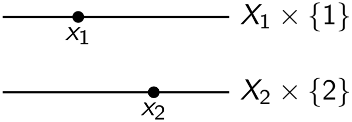

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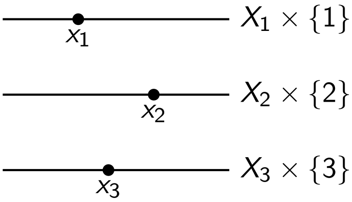

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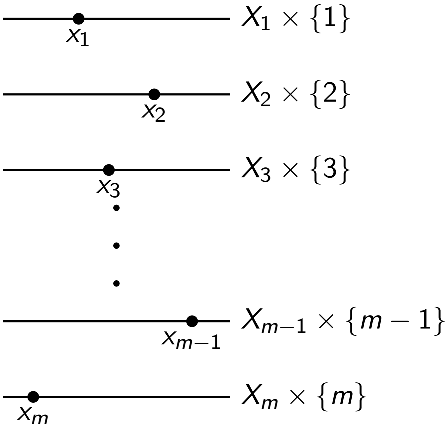

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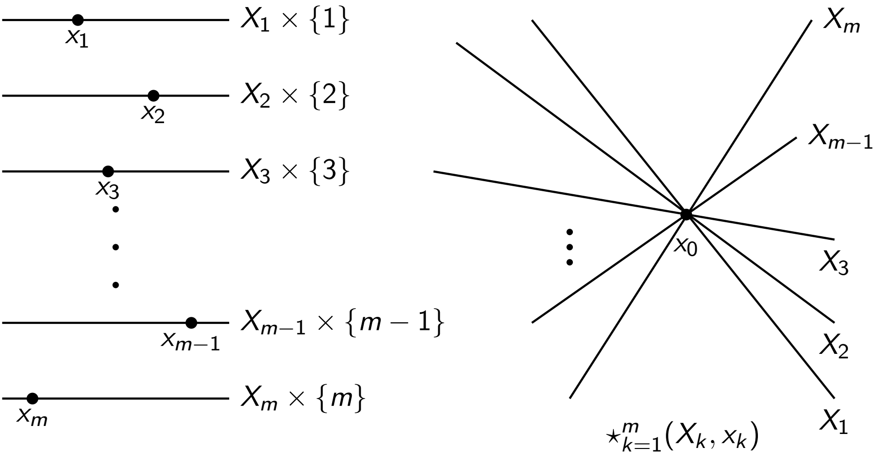

# Stars of Cook continua

## Lemma

*Let $m$ be a positive integer and for each $k \in \{1, 2, 3, \ldots, m\}$ let $X_k$ be a Cook continuum and let $x_k \in X_k$ be any point. Then $x_0$ is the only cut point for $\star_{k=1}^{m}(X_k, x_k)$.*

## Theorem

*Let $m$ be a positive integer and let $X_0$ be a Cook continuum. For each $k \in \{1, 2, 3, \ldots, m\}$, let $X_k$ be a Cook continuum, which is homeomorphic to $X_0$, let $\varphi_k : X_0 \to X_k$ be the homeomorphism from $X_0$ to $X_k$, and let $x_k \in X_k$ be any point. Also, let*

$$f : X_0 \to \star_{k=1}^{m}(X_k, x_k)$$

*be any non-constant continuous function. Then there is $k \in \{1, 2, 3, \ldots, m\}$ such that $f(x) = \varphi_k(x)$ for each $x \in X_0$.*

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*be any non-constant continuous function. Then there is $k \in \{1, 2, 3, \ldots, m\}$ such that $f(x) = \varphi_k(x)$ for each $x \in X_0$.*

### Corollary

*Let $m$ be a positive integer and let $X_0$ be a Cook continuum. For each $k \in \{1, 2, 3, \ldots, m\}$, let $X_k$ be a Cook continuum, which is homeomorphic to $X_0$, let $\varphi_k : X_0 \to X_k$ be the homeomorphism from $X_0$ to $X_k$, and let $x_k \in X_k$ be any point. Also, let*

$$f : \star_{k=1}^{m}(X_k, x_k) \to \star_{k=1}^{m}(X_k, x_k)$$

*be any non-constant continuous function. Then for each $n \in \{1, 2, 3, \ldots, m\}$ there is $k \in \{1, 2, 3, \ldots, m\}$ such that $f(x) = (\varphi_k \circ \varphi_n^{-1})(x)$ for each $x \in X_n$.*

**Theorem**

*Let $m$ be a positive integer, let $X_0$ be a Cook continuum, and let $x_0 \in X_0$ be any point. For each $k \in \{1, 2, 3, \ldots, m\}$, let $X_k$ be a Cook continuum, which is homeomorphic to $X_0$, let $\varphi_k : X_0 \to X_k$ be the homeomorphism from $X_0$ to $X_k$, and let $x_k = \varphi_k(x_0)$. Then*

$$\mathrm{degrig}(\star_{k=1}^{m}(X_k, x_k)) = \mathrm{lcm}(1, 2, \ldots, m).$$

**Theorem**

*Let $m$ be a positive integer and for each $k \in \{1, 2, 3, \ldots, m\}$ let $X_k$ be a Cook continuum and let $x_k \in X_k$ be any point such that for each $k, n \in \{1, 2, 3, \ldots, m\}$ holds that $X_k$ is homeomorphic to $X_n$ if and only if $k = n$. Also, let*

$$f : \star_{k=1}^{m}(X_k, x_k) \to \star_{k=1}^{m}(X_k, x_k)$$

*be any non-constant continuous function. Then $f = 1_{\star_{k=1}^{m}(X_k, x_k)}$. Therefore, $\star_{k=1}^{m}(X_k, x_k)$ is rigid and $\mathrm{degrig}(\star_{k=1}^{m}(X_k, x_k)) = 1$.*

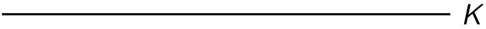
**Theorem**

*Let $X$ be a continuum such that $\mathrm{degrig}(X) \neq \infty$. For each positive integer $n$ let*

$$f_n : X \to X$$

*be any continuous surjective function. Then for each positive integer $n$ the function $f_n$ is a homeomorphism. Moreover, the inverse limit $\varprojlim\{X, f_n\}_{n=1}^{\infty}$ is homeomorphic to $X$.*

# Simple fans of Cook continua



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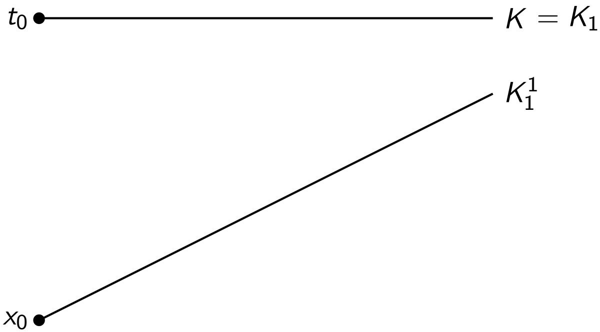

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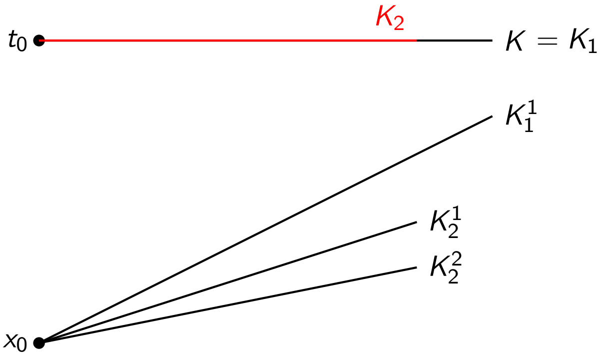

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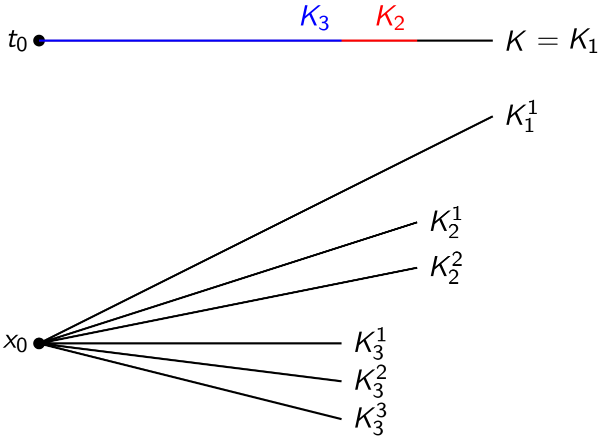

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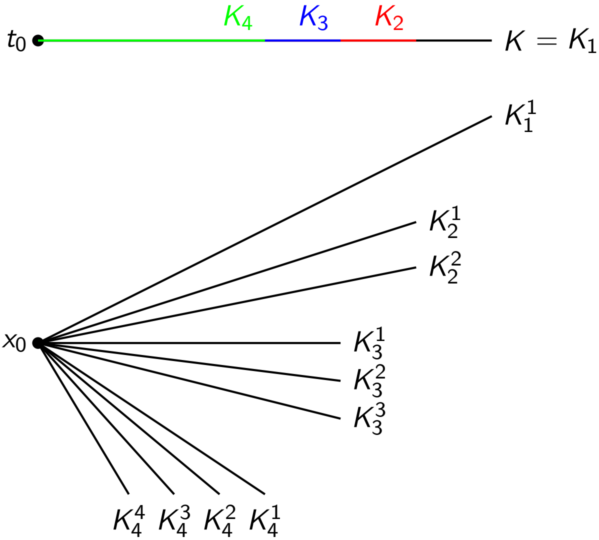

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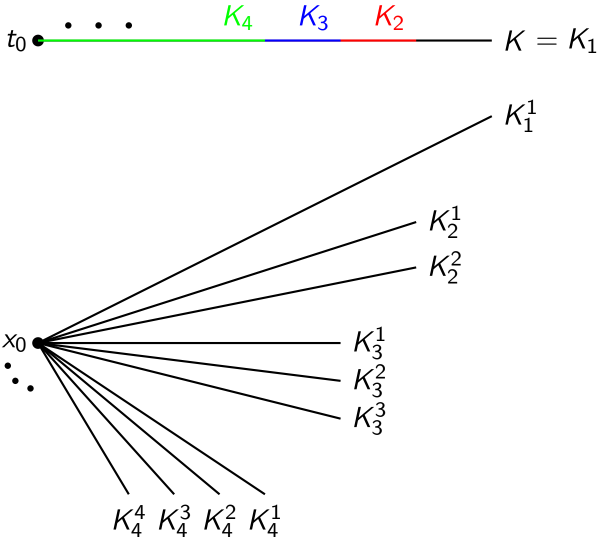

# Properties of simple fans of Cook continua

Let $X$ be a simple fan of Cook continua and let $f : X \to X$ be any surjective continuous function.

1. $x_0$ is the only cut point for $X$.
2. $f(x_0) = x_0$.
3. For each positive integer $n$ and for each $p \in \{1, 2, 3, \ldots, n\}$ there exist uniquely determined $r \in \{1, 2, 3, \ldots, n\}$ such that

$$f(K_n^p) = K_n^r.$$

4. The function $f$ is a homeomorphism.

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**Theorem**

*Let $X$ be any simple fan of Cook continua. Then $X$ is 0-rigid and* $\mathrm{degrig}(X) = \infty$.

**Theorem**

*Let $X$ be any simple fan of Cook continua. Then $\varprojlim\{X, f_n\}$ is homeomorphic to $X$ for any sequence $(f_n)_{n=1}^{\infty}$ of continuous surjective functions.*

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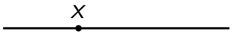
# Harmonic fan of Cook continuum

**Definition**

Let $X$ be a Cook continuum, $x_0 \in X$, and let $A = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}$. The harmonic fan of a Cook continuum $X$ is defined to be the quotient

$$(X \times A)\,|_{\sim},$$

where $\sim$ is the equivalence relation on $X \times A$, defined by $(x, a) \sim (x, a)$ for each $x \in X$ and for each $a \in A$ and $(x_0, a) \sim (x_0, b)$ for each $a, b \in A$.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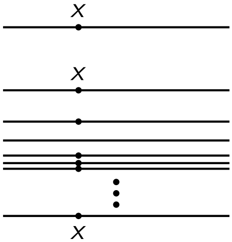
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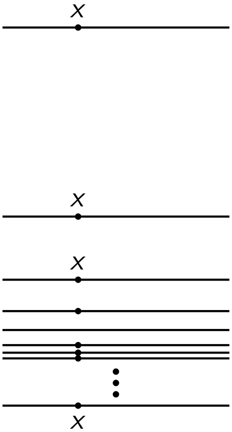
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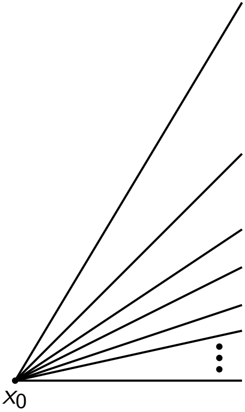

$x_0$

# Properties of harmonic fans

Let $X$ be a Cook continuum, $Y$ be a harmonic fan of Cook continuum $X$ and let $f : Y \to Y$ be a continuous surjective function.

1. $x_0$ is the only cut point for $Y$.
2. $f(x_0) = x_0$.
3. For each $a \in A$ there is $b \in A$ such that $f(X \times \{a\}) \subseteq X \times \{b\}$. Moreover, either $f(X \times \{a\})$ is homeomorphic to $X$ either $f(X \times \{a\}) = \{x_0\}$.
4. $f(X \times \{0\}) = X \times \{0\}$.
5. $Y$ is 0-rigid and $\mathrm{degrig}(Y) = \infty$.

Let $X$ be a Cook continuum and $Y$ be a harmonic fan of Cook continuum $X$. Then there is a sequence $(f_n)_{n=1}^{\infty}$ of continuous surjective functions from $Y$ to $Y$ such the the inverse limit $\varprojlim\{Y, f_n\}$ is not homeomorphic to $Y$.

# Cantor fan of Cook continuum

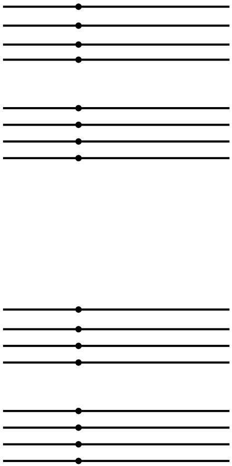
**Definition**

Let $X$ be a Cook continuum, $x_0 \in X$, and let $C$ be a Cantor set. Cantor fan of Cook continuum $X$ is defined to be the quotient

$$(X \times C)|_{\sim},$$

where $\sim$ is the equivalence relation on $X \times C$, defined by $(x, c) \sim (x, c)$ for each $x \in X$ and for each $c \in C$, and $(x_0, c) \sim (x_0, d)$ for each $c, d \in C$.

# Construction



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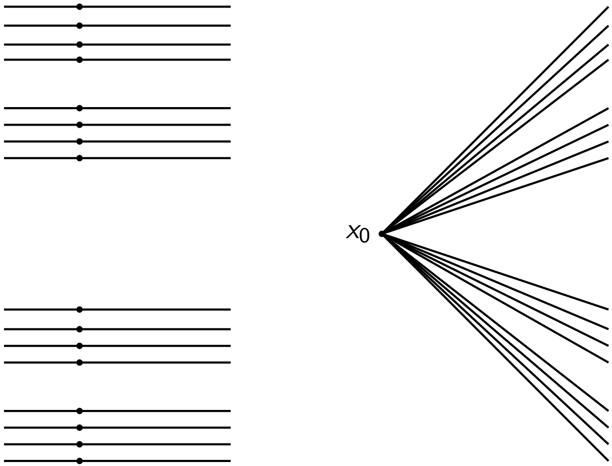

# Properties of Cantor fans

Let $X$ be a Cook continuum, $Y$ be a Cantor fan of Cook continuum $X$ and let $f : Y \to Y$ be a continuous surjective function.

1. $x_0$ is the only cut point for $Y$.
2. $f(x_0) = x_0$.
3. For each $c \in C$ there is $d \in C$ such that $f(X \times \{c\}) \subseteq X \times \{c\}$. Moreover, either $f(X \times \{c\})$ is homeomorphic to $X$ either $f(X \times \{c\}) = \{x_0\}$.
4. $Y$ is 0-rigid and $\mathrm{degrig}(Y) = \infty$.

Let $X$ be a Cook continuum and $Y$ be a Cantor fan of Cook continuum $X$. Then there is a sequence $(f_n)_{n=1}^{\infty}$ of continuous surjective functions from $Y$ to $Y$ such the the inverse limit $\varprojlim\{Y, f_n\}$ is not homeomorphic to $Y$.

# THANK YOU!