Closure Operators for Semitopogenous Spaces

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Abstract:

Semitopogenous orders on a set X were introduced by Császár to provide a unified approach to topology, proximity, and uniformity. Given a topology τ on X, one of the motivating examples is the semitopogenous order defined by $A \sqsubset U$ if and only if $A \subseteq intU$. Thus, $A \sqsubset U$ may be used to model the idea that U is a neighborhood of A. Four closure operators may now be defined from a semitopogenous order using these ideas.

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1. $x \in cl_C(A)$ iff every neighborhood of x intersects A.
2. $x \in cl_{\sqsubset}(A)$ iff every neighborhood of x is a neighborhood of some point $a \in A$.
3. A topology $\mathcal{T}_{\sqsubset}$ may be defined from $\sqsubset$ using the idea that U is open iff it is a neighborhood of each of its points, and this topology gives a Kuratowski closure operator $cl_{\mathcal{T}_{\sqsubset}}$.

4. as an analog of the kernel of $A = \bigcap\{U : A \subset U, U \text{ open}\}$, $cl^{\sqsubset} A = \bigcap\{U : A \sqsubset U\}$.

We provide a systematic comparison of these closure operators. Examples are presented to show their relative dependence and independence.

Semitopogenous Orders

Consider the following conditions which a binary relation $\sqsubset$ on the power set $\mathcal{P}(X)$ might satisfy:

(S1) $\emptyset \sqsubset \emptyset$, $X \sqsubset X$.

(S2) $A \sqsubset B$ implies $A \subseteq B$.

(S3) $A \subseteq A' \sqsubset B' \subseteq B$ implies $A \sqsubset B$.

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(S4) $A \sqsubset B$ and $A' \sqsubset B'$ implies $A \cap A' \sqsubset B \cap B'$, and (S4-$\cap$)
$A \sqsubset B$ and $A' \sqsubset B'$ implies $A \cup A' \sqsubset B \cup B'$. (S4-$\cup$)

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$A \sqsubset B$ and $A' \sqsubset B'$ implies $A \cup A' \sqsubset B \cup B'$. (S4-$\cup$)

(S5) $A \sqsubset B$ implies there exists $C \subseteq X$ with $A \sqsubset C \sqsubset B$.

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A relation satisfying (S5) is said to be *interpolating*.

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Every (semi-)topogenous order $\sqsubset$ has a **complementary order** $\sqsubset^c$**:**

$$A \sqsubset^c B \iff X \setminus B \sqsubset X \setminus A.$$

If τ is a topology on X,

$$A \sqsubset_\tau^c B \iff clA \subseteq B.$$

Closure Operators

A *closure operator* on X is a function $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ which is:

grounded: $cl\emptyset = \emptyset$,

extensive: $A \subseteq clA$ for all $A \subseteq X$,

monotone: $A \subseteq B \subseteq X \Rightarrow clA \subseteq clB$.

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If it also satisfies

additive: $cl(A \cup B) = clA \cup clB$ for all $A, B \subseteq X$, and

idempotent: $cl(cl(A)) = clA$ for all $A \subseteq X$

then it is a *Kuratowski closure operator*.

Recall that Kuratowski closure operators cl on a set X are just those closure operators that are associated with topologies τ on X.

The Closure Operators from a semitopogenous order $\sqsubset$

(1) $cl_C : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$cl_C(A) = \{x : x \sqsubset U \Rightarrow U \cap A \neq \emptyset\}$$

is grounded, extensive, and monotone.

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(2) $cl_{\sqsubset} : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

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-Richmond & Slapal (2024)

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-Maki (1986) Holgate, Iragi, Razafindrakoto (2016)

Slapal, Richmond, Iragi (2024)

(4) If $\sqsubset$ satisfies (S1), (S3), and (S4-$\cap$), then

$$\mathcal{T}_{\sqsubset} = \{U \subseteq X : x \in U \Rightarrow x \sqsubset U\}$$

is a topology on X. -Császár (2000)

The associated Kuratowski closure operator is denoted $cl_{\mathcal{T}_{\sqsubset}}$, and this is simply the topological closure from $\mathcal{T}_{\sqsubset}$.

Notation

The collection of $cl_{\sqsubset}$-closed of X is denoted $\mathcal{F}_{\sqsubset}$.

If $\mathcal{G}$ is a collection, $\mathcal{C}\mathcal{G} = \{X - G : G \in \mathcal{G}\}$.

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Theorem

Suppose $\sqsubset$ is a semitopogenous order on X.

(a) $cl_{\sqsubset} \leq cl_C \leq cl_{\mathcal{T}_{\sqsubset}}$.

(b) $\mathcal{T}_{\sqsubset} \subseteq \mathcal{C}\mathcal{F}_C \subseteq \mathcal{C}\mathcal{F}_{\sqsubset}$.

(c) $\mathcal{C}\mathcal{T}_{\sqsubset} \subseteq \mathcal{F}_C \subseteq \mathcal{F}_{\sqsubset}$.

$$cl_{\sqsubset}(A) = \{x \in X : x \sqsubset U \Rightarrow a \sqsubset U \text{ for some } a \in A\}$$

$$(a \sqsubset U \Rightarrow a \in U)$$

$$\subseteq cl_C(A) = \{x : x \sqsubset U \Rightarrow U \cap A \neq \emptyset\}$$

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The collection of $cl_{\sqsubset}$-closed subsets of X is denoted $\mathcal{F}_{\sqsubset}$.

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Proof.

We will now show $\mathcal{T}_\sqsubset \subseteq \mathcal{CF}_C$. Suppose $U \in \mathcal{T}_\sqsubset$. We want to show $X - U = cl_C(X - U)$. Suppose $y \in cl_C(X - U)$ and $y \in U$. Now $U \in \mathcal{T}_\sqsubset$ implies $y \sqsubset U$. By the definition of cl_C, $y \in cl_C(X - U)$ and $y \sqsubset U$ implies $U \cap (X - U) \neq \emptyset$, which is a contradiction. Thus, $y \in cl_C(X - U)$ implies $y \in X - U$ so $X - U$ is cl_C-closed. Thus, $\mathcal{T}_\sqsubset \subseteq \mathcal{CF}_C$.

Now we will show that $cl_C(A) \subseteq cl_{\mathcal{T}_\sqsubset}(A)$. From $\mathcal{CT}_\sqsubset \subseteq \mathcal{F}_C$ and the fact that $cl_{\mathcal{T}_\sqsubset}$ is idempotent, we have $cl_{\mathcal{T}_\sqsubset}(A) \in \mathcal{CT}_\sqsubset \subseteq \mathcal{F}_C$. $A \subseteq cl_{\mathcal{T}_\sqsubset}(A) \in \mathcal{CT}_\sqsubset \subseteq \mathcal{F}_C$ so $cl_C(A) \subseteq cl_C(cl_{\mathcal{T}_\sqsubset}(A)) = cl_{\mathcal{T}_\sqsubset}(A)$, as needed. □

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These are the only relations which always hold.

A semitopogenous order $\sqsubset_p$ from a closure operator p

Theorem

Suppose $p : \mathcal{P}(X) \to \mathcal{P}(X)$ is grounded, extensive, and monotone. For $A, B \subseteq X$, define $A \sqsubset_p B$ if and only if $p(A) \subseteq B$.

(a) *$\sqsubset_p$ is a coperfect semitopogenous order. In particular, $\mathcal{T}_{\sqsubset_p}$ is an Alexandroff topology.*

(b) *$\sqsubset_p$ satisfies (S4-$\cup$) if and only if p is additive.*

(c) *$\sqsubset_p$ satisfies (S5) if and only if p is idempotent.*

(d) *$cl^{\sqsubset_p} = p$.*

Theorem

If $\sqsubset$ is a semitopogenous order on X and $p = cl^{\sqsubset}$, then $\sqsubset \subseteq \sqsubset_p$ and $\sqsubset = \sqsubset_p$ if and only if $\sqsubset$ is coperfect.

Example 1: ε-fattening

For $\varepsilon > 0$ and $A \subseteq \mathbb{R}$, put $A_\varepsilon = \bigcup\{(a-\varepsilon, a+\varepsilon) : a \in A\}$ and

$$A \sqsubset_\varepsilon B \iff A_\varepsilon \subseteq B.$$

For $A \subseteq \mathbb{R}$, $cl_{\sqsubset_\varepsilon}(A) = A$:
Suppose $x \in cl_{\sqsubset_\varepsilon}(A)$. Now $x \sqsubset_\varepsilon (x-\varepsilon, x+\varepsilon)$.
By def of $cl_{\sqsubset_\varepsilon}$, there exists $a \in A$ with

$$\begin{aligned} a &\sqsubset_\varepsilon (x-\varepsilon, x+\varepsilon) \\ (a-\varepsilon, a+\varepsilon) &\subseteq (x-\varepsilon, x+\varepsilon) \\ x &= a \in A \end{aligned}$$

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$$\mathcal{F}_{\sqsubset_\varepsilon} = \mathcal{P}(\mathbb{R})$$

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For $A \subseteq \mathbb{R}$, $cl_C(A) = A_\varepsilon$.

$$\begin{aligned} cl_C(A) &= \{x : (x-\varepsilon, x+\varepsilon) \subseteq U \Rightarrow \exists a \in A \cap U\} \\ &= \{x : (x-\varepsilon, x+\varepsilon) \cap A \neq \emptyset\} \\ &= \{x : \exists a \in A, x \in (a-\varepsilon, a+\varepsilon)\} \\ &= \bigcup_{a \in A} (a-\varepsilon, a+\varepsilon) \\ &= A_\varepsilon. \end{aligned}$$

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Since $\sqsubset_\varepsilon$ has form $\sqsubset_p$ for the closure operator $p(A) = A_\varepsilon$, $cl^{\sqsubset_\varepsilon} = p(A) = A_\varepsilon$ by earlier theorem.

$$\mathcal{F}_\mathcal{C} = \mathcal{F}^{\sqsubset_\varepsilon} = \{\emptyset, \mathbb{R}\}.$$

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cl_C	$\geq$ Theorem $>$ $\sqsubset_\varepsilon$		$\leq$ Theorem $<$ $\sqsubset_\varepsilon$
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Example: $A \sqsubset_S B$ iff $A = \emptyset$ or $A \cup S \subseteq B$.

$$\begin{aligned} \mathcal{T}_{\sqsubset_S} &= \{U \subseteq X : x \in U \Rightarrow x \sqsubset_S U\} \\ &= \{U \subseteq X : S \subseteq U\} \cup \{\emptyset\} \\ &= Super(S). \end{aligned}$$

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It can be shown that

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cl_C(A) = cl_{\sqsubset}(A) = cl_{\mathcal{T}_{\sqsubset}}(A) = \begin{cases} A & \text{if } \; A \cap S = \emptyset \\ A \cup S & \text{if } \; A \cap S \neq \emptyset. \end{cases}
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In particular, with $X = \mathbb{R}$ and $S = [0, 1]$,

$$cl_C([2,3]) = cl_{\sqsubset}([2,3]) = cl_{\mathcal{T}_{\sqsubset}}([2,3]) = [2,3].$$

Example: $A \sqsubset_S B$ iff $A = \emptyset$ or $A \cup S \subseteq B$.

But with $X = \mathbb{R}$ and $S = [0,1]$,

$$\begin{aligned} cl^{\sqsubset_S}([2,3]) &= \bigcap\{U \subseteq X : [2,3] \sqsubset_S U\} \\ &= \bigcap\{U \subseteq X : [2,3] \cup [0,1] \subseteq U\} \\ &= [0,1] \cup [2,3] \\ &\not\subseteq [2,3] = cl_C([2,3]) = cl_{\sqsubset}([2,3]) = cl_{\mathcal{T}_{\sqsubset}}([2,3]) \end{aligned}$$

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Example: $\sqsubset_{\rightarrow}$

On $\mathbb{R}$, define $A \sqsubset B$ if and only if

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Thus the complements of $\mathcal{T}_{\sqsubset_{\rightarrow}}$-open sets are $\mathbb{R}, \emptyset$, and the sets which are bounded above and we have

$$cl_{\mathcal{T}_{\sqsubset_{\rightarrow}}}(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \mathbb{R} & \text{if } \sup A = \infty \\ A & \text{if } A \text{ is bounded above.} \end{cases}$$

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Example: $\sqsubset_{\rightarrow}$

$cl^{\sqsubset}(A) = A$ for all $A \subseteq \mathbb{R}$.
Suppose A is bounded above. For $n \in \mathbb{N}$, let $b_n = \max A + n$. Now $A \sqsubset A \cup (b_n, \infty)$, so

$$cl^{\sqsubset}(A) \subseteq \bigcap_{n \in \mathbb{N}} (A \cup (b_n, \infty)) = A.$$

Suppose A is unbounded above and a_n is a strictly increasing sequence in A diverging to ∞. Now

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If $\sup A = \infty$,

$$A = cl^{\sqsubset}(A) \subset cl_C(A) = cl_{\sqsubset_{\rightarrow}}(A) = cl_{\mathcal{T}_{\sqsubset_{\rightarrow}}}(A).$$

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It is routine to verify that for $A \subseteq \mathbb{R}$,

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$$A \subseteq B \text{ and } A \text{ is finite} \quad \text{ or } B = \mathbb{R}.$$

It is routine to verify that for $A \subseteq \mathbb{R}$,

$$cl_C(A) = cl_{\sqsubset^*}(A) = cl_{\mathcal{T}_{\sqsubset^*}}(A) = A$$

$$cl^{\sqsubset^*}(A) = \begin{cases} A & \text{if } A \text{ is finite} \\ \mathbb{R} & \text{if } A \text{ is infinite} \end{cases}$$

	$cl_{\sqsubset}$	$cl^{\sqsubset}$	$cl_{\mathcal{T}_{\sqsubset}}$
cl_C	$\geq$ Theorem $>$ $\sqsubset_{\varepsilon}$	$<$ $\sqsubset_S$ $>$ $\sqsubset_{\rightarrow}$	$\leq$ Theorem $<$ $\sqsubset_{\varepsilon}$
$cl_{\sqsubset}$	$=$	$<$ $\sqsubset_S$ $>$ $\sqsubset_{\rightarrow}$	$\leq$ Theorem $<$ $\sqsubset_{\varepsilon}$
$cl^{\sqsubset}$		$=$	$<$ $\sqsubset_{\rightarrow}$ $>$ $\sqsubset^*$

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