(Cofinitely) sensitivity for Mahavier dynamical systems

Tina Sovič

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Coauthor

- **Matevž Črepnjak** , University of Maribor

References

- I. Banič, G. Erceg, J. Kennedy and V. Nall, Chaos and mixing homeomorphisms on fans

Definitions and notation

Let X be a non-empty compact metric space and let $F \subseteq X \times X$ be a relation on X. If F is closed in $X \times X$, then we say that F is a **closed relation** on X.

Let X be a non-empty compact metric space and F a closed relation on X. **The Mahavier product of** F is defined by

$$X_F^+ = \left\{ (x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} X \mid (x_i, x_{i+1}) \in F \text{ for each positive integer } i \right\}.$$

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Let X be a non-empty compact metric space and let F be a closed relation on X. The function $\sigma_F^+ : X_F^+ \to X_F^+$, defined by

$$\sigma_F^+(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots)$$

for each $(x_1, x_2, x_3, \ldots) \in X_F^+$, is called **the shift map on X_F^+**.

Let X be a compact metric space and F a closed relation on X. The dynamical system (X_F^+, σ_F^+) is called a **Mahavier dynamical system**.

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A dynamical system (X, f) is **sensitive**, if there exists $\varepsilon > 0$ such that for each non-empty open set $U \subseteq X$, there exists a positive integer n such that $\mathrm{diam}(f^n(U)) > \varepsilon$.
In this case ε is called a **sensitivity constant** of (X, f).

Dynamical system (X, f) is **cofinitely sensitive**, if there exists $\varepsilon > 0$ such that for each non-empty open set $U \subseteq X$, there exists a positive integer n_0 such that $\mathrm{diam}(f^n(U)) > \varepsilon$ for each positive integer $n \geq n_0$.

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Example

Let $X = \mathbb{Z} \cup \{-\infty, \infty\}$. For each integer k let

$$q_k = \begin{cases} 1 - \frac{1}{2^k} & ; \quad k \in \mathbb{N} \\ 2^k - 1 & ; \quad k \in \mathbb{Z} \setminus \mathbb{N} \end{cases},$$

and let $Y = \{-1, 1\} \cup \bigcup_{k \in \mathbb{Z}} \{q_k\}$. Note that X and Y are homeomorphic. Let $h : Y \to X$ be a homeomorphism with $h(q_k) = k$ for each integer k. On X we use the metric d defined by

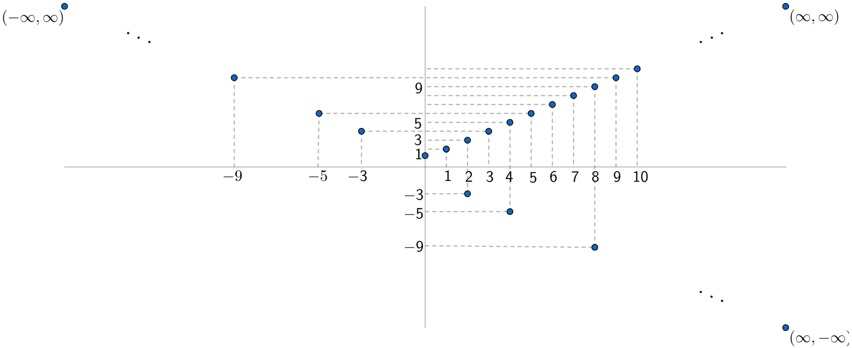
$$d(x, y) = |h^{-1}(x) - h^{-1}(y)|.$$

Example

We define a close relation F on X by

$$F = \left\{(2^k, 2^k+1), (2^k, -2^k-1), (-2^k-1, 2^k+2) \mid k \in \mathbb{N}\right\}$$
$$\cup \left\{(k, k+1) \mid k \in \mathbb{N} \setminus \{2^k \mid k \in \mathbb{N}\}\right\}$$
$$\cup \{(0,1), (\infty, \infty), (\infty, -\infty), (-\infty, \infty)\}.$$

Then (X_F^+, σ_F^+) is a sensitive dynamical system, which is not cofinitely sensitive.


$(-\infty, \infty)$
(∞, ∞)
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9
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1
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-5
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1 2 3 4 5 6 7 8 9 10
-3
-5
-9

Closed relations that imply sensitivity

Observation

Let X be a compact metric space and let F be a closed relation on X. If there exist closed relations A and B on X, with $F = A \cup B$, $A \cap B = \emptyset$ and $\pi_1(A) = \pi_1(B) = X$, then (X_F^+, σ_F^+) is a cofinitely sensitive dynamical system with $\varepsilon = \min\{d(A(x), B(x)) \mid x \in X\} > 0$ being its sensitivity constant.

Theorem

Let X be a compact subspace of $\mathbb{R}$ and let F be a closed relation on X. If there exist closed relations A and B on X, such that

1. $F = A \cup B$,
2. $A \cap B \neq F$,
3. $\pi_1(A) = \pi_1(B) = X$,
4. $(A \cap B) \cap \Delta = \emptyset$, *and*
5. $A \subseteq \{(x, y) \in X \times X \mid y \leq x\}$ *or* $A \subseteq \{(x, y) \in X \times X \mid y \geq x\}$,

then (X_F^+, σ_F^+) *is sensitive.*

Example

Let $X = [0,1]$ and let $F \subseteq X \times X$ be the union of two straight line segments, first one from $(1,0)$ to $(0,\frac{1}{2})$ and the second one from $(0,\frac{1}{2})$ to $(1,1)$. Then (X_F^+, σ_F^+) is sensitive.

Theorem

Let X be a compact subspace of $\mathbb{R}$ with $m = \min X$ and $M = \max X$. Let F be a closed relation on X, satisfying

1. $\pi_2(F) \subseteq \pi_1(F)$,
2. *for each $x \in X$ it holds that $m \in F(x)$ if and only if $x = m$,*
3. *there exist closed relations A and B on X such that*
    1. $F = A \cup B$,
    2. $A \subseteq \{(x,y) \in X \times X \mid y \leq x\}$, $B \subseteq \{(x,y) \in X \times X \mid y \geq x\}$,
    3. $A \cap \Delta = \{(m,m)\}$, $B \cap \Delta = \{(m,m)\}$,
    4. *there exist $a \in (m,M)$ with $[m,a] \subseteq \pi_1(A) \cap \pi_1(B)$.*

Then (X_F^+, σ_F^+) is cofinitely sensitive.

Example

Let $X = [0,1]$ and $F = \{(t, \frac{t}{2}) \mid t \in [0,1]\} \cup \{(t, \sqrt{3}t) \mid t \in [0, \frac{1}{\sqrt{3}}]\}$. Then X_F^+ is homeomorphic to the Lelek fan, and (X_F^+, σ_F^+) is cofinitely sensitive.

Theorem

Let $a, b \in \mathbb{R}$, $a < b$, $X = [a, b]$, *let* $f : X \to X$ *be a strictly increasing continuous function, and let* B *be a closed relation on* $Y \subseteq X$, *such that for each* $x \in X$ *with* $f(x) = x$, *there exist* $a_x, b_x \in Y$, $a_x < b_x$, *such that*

1. $x \in [a_x, b_x] \subseteq Y$ *and*
2. $\Gamma(f) \cap B|_{[a_x, b_x]}$ *is an empty set.*

If $F = \Gamma(f) \cup B$, *then* (X_F^+, σ_F^+) *is sensitive.*

Theorem

Let $a, b \in \mathbb{R}$, $a < b$, $X = [a, b]$, let $f : X \to X$ be a strictly decreasing continuous function, and let B be a closed relation on $Y \subseteq X$, such that for each $x \in [a, b]$ with $f^2(x) = x$, there exist $a_x, b_x \in Y$, $a_x < b_x$, such that

1. $x \in [a_x, b_x] \subseteq Y$ *or* $f(x) \in [a_x, b_x] \subseteq Y$ *and*
2. $\Gamma(f) \cap B|_{[a_x, b_x]}$ *is an empty set.*

If $F = \Gamma(f) \cup B$, then (X_F^+, σ_F^+) is sensitive.

Subsets of a closed relation and sensitivity

Theorem

Let X be a compact metric space and G a closed relation on X such that (X_G^+, σ_G^+) is sensitive. Suppose F is a closed relation on X with $G \subseteq F$, $\pi_2(F) \subseteq \pi_1(G)$, and σ_F^+ is an open map. Then (X_F^+, σ_F^+) is sensitive, too.

Observation

Let X be a compact metric space, let n be a positive integer and let $F = \bigcup_{i=1}^{n} \Gamma(h_i)$, where $h_i : X \to X$ is a homeomorphism for each i. Then $\sigma_F^+ : X_F^+ \to X_F^+$ is an open map.

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For a compact metric space X, let

$$\Delta_X = \{(x,x) \mid x \in X\}.$$

Theorem

Let X be a compact metric space and let F be a closed relation on X. If $\Delta_X \subseteq F$ and (X_F^+, σ_F^+) is sensitive, then (X_F^+, σ_F^+) is cofinitely sensitive.

Theorem

Let X be a compact metric space, let G be a closed relation on X with $\pi_1(G) = \pi_2(G) = X$. If $F = G \cup \Delta_X$ and (X_G^+, σ_G^+) is sensitive, then (X_F^+, σ_F^+) is cofinitely sensitive.

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Thank you !

tina.sovic@um.si