Filters on ω and convergence of measures on Boolean algebras

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A sequence of measures $\langle \mu_n\colon n \in \omega \rangle$ on $\mathcal{A}$ is

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Nikodym property of Boolean algebra

A Boolean algebra $\mathcal{A}$ has the *Nikodym property* if every pointwise null sequence of measures on $\mathcal{A}$ is uniformly bounded.

The Nikodym property - examples

Examples of Boolean algebras with the Nikodym property

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if $x_n \to x$, then consider the sequence of measures $\mu_n = n(\delta_{x_n} - \delta_x)$.

N_F spaces

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F - a free filter on ω

$N_F = \omega \cup \{p_F\}$, where $p_F \notin \omega$, with the following topology:

- every point of ω is isolated in N_F,
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Question

For which filters F on ω, if N_F embeds into the Stone space $St(\mathcal{A})$ of a Boolean algera $\mathcal{A}$, then $\mathcal{A}$ does not have the Nikodym property?

Finitely supported AN-sequences

A Borel measure μ on a topological space X is *finitely supported* if $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and distinct $x_1, \ldots, x_n \in X$.

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$\mathcal{AN}$ - the class of all ideals on ω whose dual filters do not have the Nikodym property

Destroying the Nikodym property

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Theorem

If N_F homeomorphically embeds into $St(\mathcal{A})$ and F does not have the Nikodym property, then $\mathcal{A}$ does not have the Nikodym property.

Proposition

If a filter F on ω does not have the Nikodym property, then there is an fs-AN sequence of measures (μ_n) on N_F which is disjointly supported, that is, $supp(\mu_k) \cap supp(\mu_l) = \varnothing$ for every $k \neq l \in \omega$.

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A filter F on ω has the Nikodym property if and only if there is no sequence $\langle \mu_n : n \in \omega \rangle$ of non-negative measures on ω with finite disjoint supports such that:

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- $\lim_{n\to\infty} \mu_n(\omega) = \infty$,
- $\lim_{n\to\infty} \mu_n(\omega \setminus A) = 0$ *for every* $A \in F$.

Definition

$\varphi\colon \mathcal{P}(\omega) \to [0, \infty]$ is a *submeasure* if

- $\varphi(\varnothing) = 0$ and $\varphi(\{n\}) < \infty$ for every $n \in \omega$,
- $\varphi(X) \le \varphi(Y)$ whenever $X \subseteq Y$,
- $\varphi(X \cup Y) \le \varphi(X) + \varphi(Y)$ for every $X, Y \subseteq \omega$.

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Exhaustive ideal of submeasure φ

The following ideal on ω is called *the exhaustive ideal* of φ:

$$Exh(\varphi) = \left\{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus [0, n]) = 0\right\}$$

It is always an $F_{\sigma\delta}$ P-ideal.

Special types of submeasures

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Non-pathological submeasure

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An example is an *asymptotic density* on ω defined by:

$$\varphi_d(A) = \sup_{n \in \omega} \left| A \cap [2^n, 2^{n+1}) \right| / 2^n.$$

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The Nikodym property and submeasures

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Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. We write that $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f\colon \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$ for all $I \in \mathcal{I}$.

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There exists a family $\{\mathcal{J}_\alpha\colon \alpha < \mathfrak{d}\}$ of density ideals from $\mathcal{AN}$ such that for any $\mathcal{I} \in \mathcal{AN}$ there is $\alpha < \mathfrak{d}$ such that $\mathcal{I} \leq_K \mathcal{J}_\alpha$.

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- (Głodkowski & Widz, preprint 2024) forcing construction of the algebra with (G) but without (N)

Theorem (Marciszewski, Sobota)

Let F be a filter on ω and $\mathcal{A}$ a Boolean algebra. Then,

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For a density ideal $\mathcal{I}$ on ω we have $\mathcal{I} \in \mathcal{AN}$ if and only if $\mathcal{I} <_K \mathcal{Z}$.

Boolean algebras associated to filters

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Corollary

There are $\mathfrak{c}$ *non-isomorphic Boolean algebras with* (N) *and without* (G), *of the form* $\mathcal{A}_F$ *where* F *is a Borel filter on* ω.

The end

Thank You! :)