Hyperspace Shadowing

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Summer Topology Conference, Coimbra 2024

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General Set Up

Let X be a compact metric space and $T : X \to X$ be a homeomorphism. We call (X, T) a TDS, i.e., a topological dynamical system.

Let 2^X be space of non-empty compact subsets of X endowed with the Hausdorff metric.

$2^T : 2^X \to 2^X$ is the map which takes set K to set $T(K)$.

$C(X)$ is the subspace of 2^X consisting of compact connected subsets (i.e., continua) of X.

$C(T) : C(X) \to C(X)$ is the map which takes set K to set $T(K)$.

Basic Problem

Question

How are dynamics of T, 2^T and $C(T)$ are related?

In this talk we focus on the shadowing property.

Definition

Let (X, T) be a TDS.

- $\{x_n\}_{n\in\mathbb{Z}}$ is a δ-pseudotrajectory means that $d(T(x_n), x_{n+1}) < \delta$ for all $n \in \mathbb{Z}$.
- point x ε-shadows $\{x_n\}_{n\in\mathbb{Z}}$ means that $d(T^n(x), x_n) < \varepsilon$ for all $n \in \mathbb{Z}$.
- (X, T) has the shadowing property means that for all $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudotrajectory $\{x_n\}$ is ε-shadowed by some $x \in X$.

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Main Problem of this talk

Problem

Let (X, f) be a TDS with the shadowing property. When does $C(f)$ has the shadowing property?

Some Known Results

Theorem (Fernandez-Good, Fund Math 2016)

T has the shadowing property if and only if 2^T has the shadowing property.

Let $F_\infty(X)$ be the set of all finite subset of X. They prove that T having the shadowing property implies that 2^T restricted to $F_\infty(X)$ has the finite shadowing property.

Known results, cont'd

Theorem (Arbeito-Bohorquez, Math Zeit, 2023)

Let $T :: \mathbb{S}^1 \to \mathbb{S}^1$ be a Morse-Smale diffeomorphism. Then T is has the shadowing property but $C(T)$ does not.

In particular, consider $T :: \mathbb{S}^1 \to \mathbb{S}^1$ which has exactly two fixed points, one attracting and the other repelling. Then, $C(T)$ does have the shadowing property.

Expansive, Continnuum-wise Expansive

Definition

For each $x \in X$ and $c > 0$, let

$$W^s_c(x) := \{y \in X;\ d(f^k(y), f^k(x))\ \ c \text{ for every } k >= 0\}$$

be the *c-stable set* of x and

$$W^u_c(x) := \{y \in X;\ d(f^k(y), f^k(x))\ \ c \text{ for every } k <= 0\}$$

be the *c-unstable set* of x. The dynamical ball of x with radius c is the set

$$\Gamma_c(x) = W^u_c(x) \cap W^s_c(x).$$

We say that f is continnuuum-wise expansive (for short, cw-expansive) if there exists $c > 0$ such that $\Gamma_c(x)$ is totally disconnected for every $x \in X$.

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Stable and Unstable Continua

Definition

For each $x \in X$ and $c > 0$, denote by $C_c^s(x)$ the c-stable continuum of x, i.e., the connected component of x in $W_c^s(x)$,

Let $C_c^u(x)$ the c-unstable continuum of x, be the connected component of x in $W_c^u(x)$.

$$\mathscr{C}^s = \{C \in \mathscr{C}(X) : \operatorname{diam}(f^n(C)) \to 0 \text{ when } n \to +\infty\} \text{ and}$$

$$\mathscr{C}^u = \{C \in \mathscr{C}(X) : \operatorname{diam}(f^{-n}(C)) \to 0 \text{ when } n \to \infty\}.$$

Continua in $\mathscr{C}^s$ are called stable and continua in $\mathscr{C}^u$ are called unstable.

ε-stable and unstable continua

Definition

Let

$$\mathscr{C}^s_\varepsilon = \{C \in \mathscr{C}(X) \,:\, \mathrm{diam}(f^n(C)) <= \varepsilon \text{ for every } n >= 0\} \text{ and}$$

$$\mathscr{C}^u_\varepsilon = \{C \in \mathscr{C}(X) \,:\, \mathrm{diam}(f^{-n}(C)) <= \varepsilon \text{ for every } n >= 0\}.$$

These sets contain exactly the ε-stable and ε-unstable continua of f, respectively.

Theorem (Kato, Cand. J Math, 1993)

If $c > 0$ is a cw-expansive constant of f and $\varepsilon < \frac{c}{2}$, then

$$\mathscr{C}^s_\varepsilon \subset \mathscr{C}^s \quad \textit{and} \quad \mathscr{C}^u_\varepsilon \subset \mathscr{C}^u.$$

Negative Results

Theorem (Carvalho-D)

Let $f : X \to X$ be a cw-expansive homeomorphism of a compact metric space (X, d). If for each $n \in \mathbb{N}$, there is a pair of non-trivial stable/unstable continua (S_n, U_n) satisfying

1. *there exists $r > 0$ such that* $\operatorname{diam}(S_n) > r$ *and* $\operatorname{diam}(U_n) > r$ *for every* $n \in \mathbb{N}$,
2. $d_H(S_n, U_n) \to 0$ *as* $n \to +\infty$ *and*

then $C(f)$ does not have the shadowing property.

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Negative result's continued

The following definition was introduced by Artigue-Carvalho-Cordeiro-Vieitez in 2024.

Definition

We say that f satisfies the *cw*-local-product-structure if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$C^s_\varepsilon(x) \cap C^u_\varepsilon(y) \neq \emptyset \quad \text{whenever} \quad d(x,y) < \delta.$$

The *cw*-expansive homeomorphisms satisfying the *cw*-local-product-structure are called *cw*-hyperbolic.

Theorem (Carvalho-D)

If $f : X \to X$ is a transitive cw-hyperbolic homeomorphism, then $C(f)$ does not have the shadowing property. In particular, if $f : M \to M$ is a transitive Anosov diffeomorphism, then $C(f)$ does not have the shadowing property.

Dendrites

Definition

A dendrite is a locally connected continuum which contains no simple closed curve.

In some loose sense, dendrites can be thought of trees which are allowed infinite tentacles.

Example

Arcs, trees, D_n, D_∞.

Some known results on dendrites

Theorem (Brian-Meddaugh-Raines, Discrete Cont Dyn S, 2019)

Let D be a dendrite. Then, a generic continuous surjection on D has the shadowing property.

What about homeomorphisms? Pennings-Eeuwen [Real Anal Exch 1990] gave a necessary and sufficient condition for a homeomorphism of the interval to have the shadowing property.

Theorem (Artigue Cousillas, Axioms, 2019)

The space of order-preserving homeomorphisms of the interval $[0, 1]$ *has a comeager conjugacy class and this map has the shadowing property. There exists a planer one-dimensional continuum (not a dendrite) whose homemorphism group has a comeager conjugacy class and this homeomorphism has the shadowing property.*

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Homeomorphism Groups of D_n, D_∞.

Using some deep techniques from projective Fraïssé limit from descriptive set theory, Duchesne proved the following.

Theorem (Duchesne, J Ecol Poly Math, 2020)

The group of homeomorphisms of D_∞ has a comeage conjugacy class. None of the group of homemorphisms of D_n, $2 < n < \infty$ has a dense conjugacy class.

Dendrites and the Shadowing Property

Example

There is a dendrite which admits no homemorphism with the shadowing property.

The above follows from the fact that there is non-trivial dendrite whose only homemorphism is the identity map.

Theorem (Carvalho-D)

Fix $n \in \{3, 4, \dots, \infty\}$. Let D be a dendrite such that all branch points of D have order n. Then, D admits a homeomorphism with the shadowing property.

Dendrite and the Shadowing Property for $C(f)$.

Definition

A mapping $f : X \to X$ is said to be monotone if $f^{-1}(p)$ is connected for all $p \in X$. Clearly, every homeomorphism is a monotone map.

Theorem (Carvalho-D)

Let D be a dendrite and $f : D \to D$ be a monotone map. If f has the shadowing property, then so does $C(f)$.

Muito Obrigado!