Classification of trees that inscribe hyperbolic ε-rectangles

38th SUMMER CONFERENCE ON TOPOLOGY AND ITS APPLICATIONS

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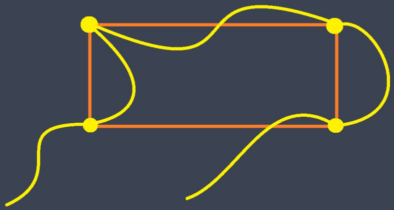
» The Square Peg Problem

A subset X of the plane admits an inscribed polygon, P, if all vertices of a polygon similar to P lie on X.

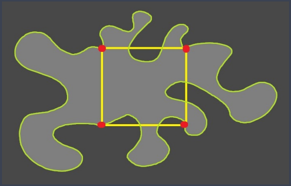


Orange rectangle inscribed in a yellow arc.

In 1911 Otto Toeplitz asked:

¿Does every Jordan curve admit an inscribed square?

This is known as the square peg problem (or the inscribed square problem).

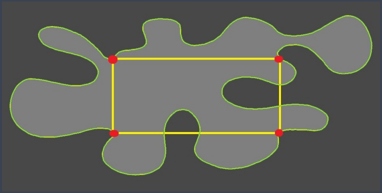


square inscribed in a Jordan curve

» Resolved cases

* Piecewise analytic curves
  Arnold Emch, 1916, [3]
* Locally monotone curves
  Walter R. Stromquist, 1989, [10]
* Symmetric curves
  Mark J. Nielsen and Stephen E. Wright, 1995, [9]
* Curves that lie (homotopically essential) in an annulus whose outer radius is at most $1 + \sqrt{2}$ times its inner radius
  Benjamin Matschke, 2011, [6]
* Curves formed by the union of the graphs of two Lipschitz continuous functions
  Terence Tao, 2017, [11]

» The Inscribed Rectangle Problem



Rectangle inscribed in a Jordan curve

A natural variant of the square peg problem is: Does every Jordan curve admit an inscribed rectangle? The answer is affirmative (H. Vaughan 1977, [7, p.71]).

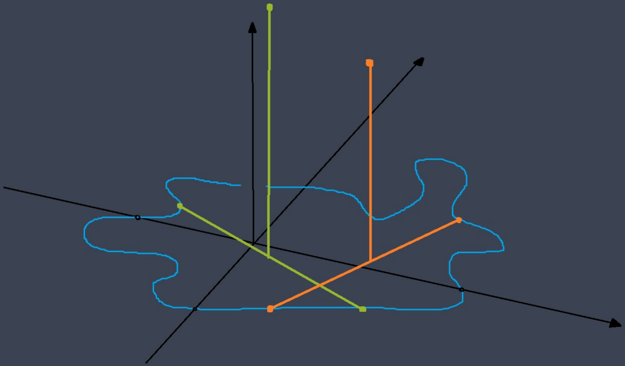
» The solution of H. Vaughan

Let $\gamma : S^1 \to \mathbb{R}^2$ be a Jordan Curve. For each set $\{a, b\}$, with $a, b \in S^1$, define the function:

Vaughan's function, f_γ

$$f_\gamma(\{a, b\}) = \begin{cases} \left(\frac{\gamma(a)+\gamma(b)}{2}, \|\gamma(a) - \gamma(b)\|\right), & \text{if } a \neq b; \\ (\gamma(a), 0), & \text{if } a = b. \end{cases}$$

If there are 4 different points $a, b, c, d \in S^1$ such that $f_\gamma(\{a, b\}) = f_\gamma(\{c, d\}) \Rightarrow \{\gamma(a), \gamma(b), \gamma(c), \gamma(d)\}$ are the vertices of a rectangle inscribed in $\gamma(S^1)$.



» The solution of H. Vaughan

f_γ is not one to one

2^{nd} symmetric product, F_2

Given a continuum X,

$$F_2(X) = \{A \subseteq X : A \neq \emptyset \text{ y } A \text{ has at most 2 points}\}$$

topologized by the Hausdorff metric.

We can see that:

* f_γ is a continuous function from $F_2(S^1)$ to $\mathbb{R}^3$,
* $F_2(S^1)$ is homeomorphic to the Möbius strip,
* If f_γ is one to one $\Rightarrow \exists$ an embedding from $\mathbb{P}_2(\mathbb{R})$ to $\mathbb{R}^3$ (Schoenflies).

$\therefore$ f_γ is not one to one.

** Watch this video of 3Blue1Brown: Who cares about topology?

» Other Generalizations

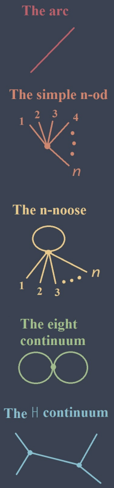
* Can something be said about the ratio between the sides of the inscribed rectangles?
  * In 2020 Joshua Evan Greene and Andrew Lobb proved that every smooth Jordan curve inscribes at least one rectangle of any given ratio, in particular a square (Annals of Mathematics (2) 194, No. 2, 509-517 (2021)). The proof relies on the theorem of Shevchishin and Nemirovski that the Klein bottle does not admit a smooth Lagrangian embedding in $\mathbb{C}^2$.
  * Later they proved that every smooth Jordan curve inscribes every cyclic quadrilateral. The proof relies on the theorem of Polterovich and Viterbo that an embedded Lagrangian torus in $\mathbb{C}^2$ has minimum Maslov number 2.

* We generalized Vaughan's result to other plane sets (not necessarily Jordan curves)
  * We classify locally connected plane continua that inscribe rectangles.
  * We show that every copy of two disjoint simple triods always inscribe a rectangle.
  * We show that a dense union of disjoint arcs, such that one of them is a line segment, always inscribes a rectangle.
  * We switch to the hyperbolic plane and prove results concerning hyperbolic quadrilaterals.

» Locally Connected Plane Continua

Some locally connected plane continua are:

- The arc = The space homeomorphic to the closed interval $[0, 1]$
- The simple n-od = graph with only one ramification point, n final points and without cycles.
- The n-noose = graph homeomorphic to a circle, $\mathcal{C}$, attached to n arcs, $A_1, \ldots, A_n$, so that $\exists p \in \mathcal{C}$ s.t. $\forall i$, $\mathcal{C} \cap A_i = \{p\}$.
- The eight continuum = two joined circumferences that intersect each other in a single point.
- The H continuum = the continuum homeomorphic to the symbol H.

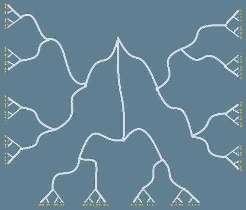


» ¿Which ones inscribe rectangles?

A plane continuum X inscribes rectangles if for all embedding $\gamma : X \to \mathbb{R}^2$ all the vertices of an euclidean rectangle lie on $\gamma(X)$.

Examples

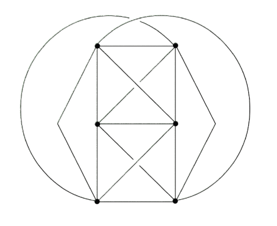
These embeddings, of the arc and the 3-od, do not inscribe rectangles.



This is a locally connected plane continua containing an infinite amount of 3-ods and the Cantor set.
¿Does it inscribe rectangles?

» Intrinsically linked graphs

A graph is intrinsically linked if any embedding of it in $\mathbb{R}^3$ contains a nontrivial link.



Conway and Gordon proved that K_6 is intrinsically linked. The proof is based on a parity argument.

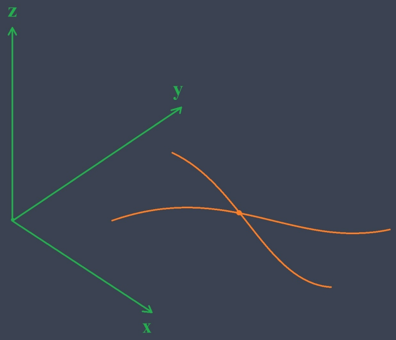
$$\lambda(X) = \sum_{\{C_1, C_2\}} lk(C_1, C_2) = 1(mod2)$$

» The case of the 4-od

Let X be a simple 4-od and $\gamma : X \to \mathbb{R}^2$ an embedding. Using the function $f_\gamma : F_2(X) \to \mathbb{R}^3$ we can see that:

Proposition

If X does not inscribe rectangles, we can define an embedding of the cone of the graph K_5 in $\mathbb{R}^3$.



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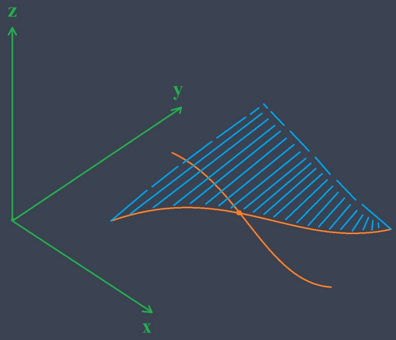

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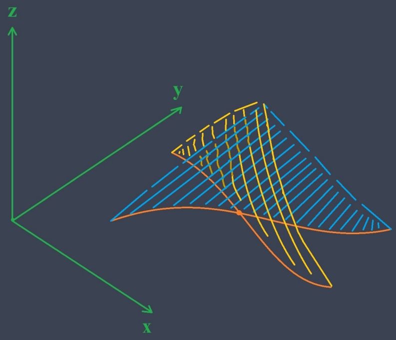

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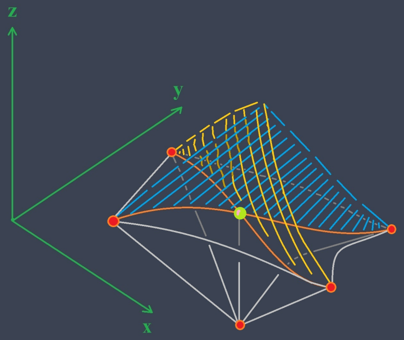


» The case of the 4-od

Let X be a simple 4-od and $\gamma : X \to \mathbb{R}^2$ an embedding. Using the function $f_\gamma : F_2(X) \to \mathbb{R}^3$ we can see that:

Proposition

If X does not inscribe rectangles, we can define an embedding of the cone of the graph K_5 in $\mathbb{R}^3$.

However, Conway and Gordon proved that every embedding of the graph K_6 in $\mathbb{R}^3$ contains a non-trivial link. Using this fact, Castañeda proved that the cone of K_5 is not embeddable in $\mathbb{R}^3$. Therefore f_γ is not one to one. So, the simple 4-od inscribes rectangles.

Lemma (M, Villanueva) [8]

The simple 4-od inscribes rectangles.

» Characterisation

Sachs proved that every embbeding of the graph $K_{3,3,1}$ in $\mathbb{R}^3$ contains a non-trivial link. Using this fact and Conway and Gordon's result Castañeda classified locally connected continua whose second symmetric product is embeddable in the $\mathbb{R}^3$:

Theorem, Castañeda, [1]

Let X be a locally connected continuum.
$F_2(X)$ can be embedded in $\mathbb{R}^3 \iff X$ is homeomorphic to: the arc, the simple 3-od, the simple 4-od, S^1, the 1-noose, the 2-noose or the eight continuum.

So we have:

* The 3-od and the arc do not inscribe rectangles.
* The space S^1 and the 4-od do inscribe rectangles.
* If X does not inscribe rectangules $\Rightarrow$ f_γ is an embedding of $F_2(X)$ in $\mathbb{R}^3$.

Theorem (M, Villanueva) [8]

The only locally connected plane continua that do not inscribe rectangles are the arc and the simple 3-od.

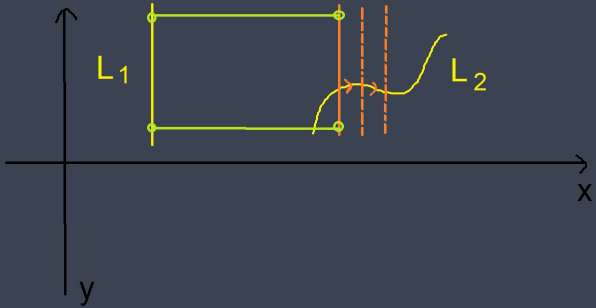
Theorem (M, Villanueva)

Let X be the disjoint union of two simple 3-ods. Then, X admits an inscribed rectangle.

Proof: $T_1 \times T_2$ is homeomorphic to the cone of the bipartite graph $K_{3,3}$ (Castañeda's result). So, if $\gamma(X)$ does not inscribe a rectangle f_γ is an embedding of cone of the bipartite graph $K_{3,3}$ in $\mathbb{R}^3$, but Castañeda proved that this is impossible. Therefore, f_γ is not injective, hence X admits an inscribed rectangle.

Theorem (M, Villanueva)

Let X be the disjoint union of a dense set of arcs in $\mathbb{R}^2$, such that at least one of them is a line segment, then X inscribes a rectangle.



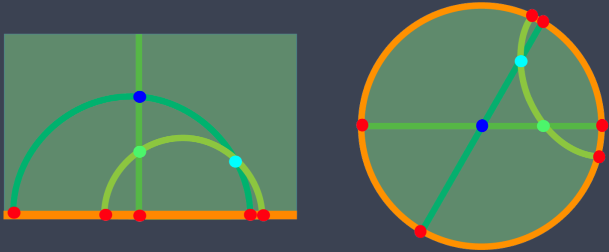
Let us define the *hyperbolic length* in $\mathbb{H}$ by the formula

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{|dz|^2}{y^2} \qquad (z = x + iy).$$

More precisely, if $\gamma : I \to \mathbb{H}$ is a piecewise differentiable path with $\gamma(t) = x(t) + iy(t) = z(t)$, then its hyperbolic length $h(\gamma)$ is equal to

$$h(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}{y} = \int_0^1 \frac{\left|\frac{dz}{dt}\right| dt}{y}.$$

With this length, it is possible to show that for any two points in $\mathbb{H}$ there is a unique path of shortest hyperbolic length, such paths are called *hyperbolic line segments* or h-segments. Once defined the hyperbolic metric on $\mathbb{H}$, we can pass to the model of the Poincare disk using the Cayley transformation $C(z) = \frac{z-i}{z+i}$.



Let X be a plane continuum and let $\gamma : X \to \mathbb{H}$ be an embedding. We define the continuous functions

$$f_\gamma : F_2(X) \to \mathbb{H} \times \mathbb{R}$$

by

$$f_\gamma(\{a, b\}) = \begin{cases} (\mathbf{midpoint}_{\mathbb{H}}(\gamma(a), \gamma(b)), d_{\mathbb{H}}(\gamma(a), \gamma(b))\,, & \text{if } a \neq b; \\ (\gamma(a), 0)\,, & \text{if } a = b. \end{cases}$$

$$g_\gamma : X \times X \to \mathbb{H} \times \mathbb{R}$$

by

$$g_\gamma((a, b)) = (\mathbf{midpoint}_{\mathbb{H}}(\gamma(a), \gamma(b)), d_{\mathbb{H}}(\gamma(a), \gamma(b))).$$

Theorem (Gauss Bonnet)

Let $\triangle$ be a hyperbolic triangle with angles α, β γ. Then the hyperbolic area $\mu(\triangle)$ is given by: $\mu(\triangle) = \pi - \alpha - \beta - \gamma$.

Definition:

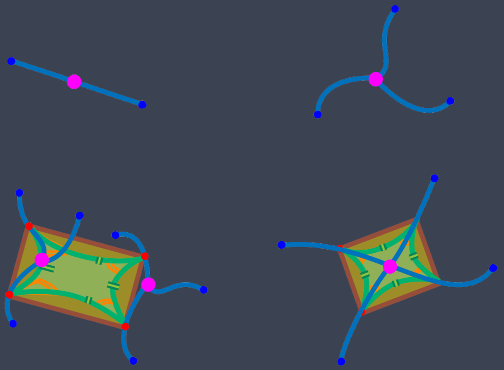
Let $\varepsilon > 0$. An ε-rectangle in $\mathbb{H}$ is a quadrilateral whose diagonals have the same hyperbolic length, the diagonals share their midpoint and the quadrilateral's inner angles sum up more than $2\pi - \varepsilon$

Definition

A topological set, X, quasi-inscribes rectangles in $\mathbb{H}$, if for every topological copy of X in $\mathbb{H}$ and every $\varepsilon > 0$, X admits inscribed an ε-rectangle.

Theorem (M, Díaz, Valdez)

The only trees that does not quasi-inscribes rectangles in $\mathbb{H}$ are the Arc and the 3-Star.



» Further questions and Ongoing work

- Do non-unicoherent plane continua inscribe rectangles (squares)?
- A. Illanes presented a family of nonlocally connected continua whose second symmetric product is embeddable in $\mathbb{R}^3$ [4]; Can we say something about this family?, Do Nonlocally connected continua inscribe rectangles (squares)?
- Can Greene-Lobb's result be generalized to more complicated continua. For instance: Does every "Smooth" embedding of the Warsaw circle in $\mathbb{R}^2$ inscribe a rectangle of any ratio?
- Does every Jordan Curve inscribe a square (a rectangle of any ratio)?
- Does Jordan curves in $\mathbb{H}$ admit ε-rectangles inscribed for every $\varepsilon > 0$.

» References

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Thank you!