Menger and consonant spaces in the Sacks model

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Convention

All spaces are metrizable, separable and zero-dimensional, i.e., subspaces of 2^ω up to homeomorphisms.

A space is totally imperfect, if it contains no copy of 2^ω.

Menger spaces and relatives

A topological space X is *Menger* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a cover of X.

A topological space X is *Hurewicz* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X.

$\mathcal{U}$ is a γ-cover of X if $\forall x \in X \forall^* U \in \mathcal{U} (x \in U)$.

σ-compact $\rightarrow$ Hurewicz $\rightarrow$ Menger $\rightarrow$ Lindelöf.

Theorem (Bartoszyński, Tsaban, 2006)

There exists a totally imperfect Menger space of size $\mathfrak{d}$.

It is natural to ask:

Question

Does there exist a totally imperfect Menger space of size $\mathfrak{c}$ if $\mathfrak{d} < \mathfrak{c}$?

Our Answer: This is independent from ZFC.

For the **existence**: Consider a finite support iteration of Hechler forcing over a model of $\neg$CH of length ω_1. Then $2^\omega \cap V$ is totally imperfect Hurewicz and of size $\mathfrak{c}$.

For the **non-existence**: the Sacks model.

Sacks model

Definition

$T \subseteq 2^{<\omega}$ is a Sacks tree if T is closed under initial segments; and for every $t \in T$ there is $s \supseteq t$ such that $s^\frown 0 \in T$ and $s^\frown 1 \in T$.

Sacks forcing: $\mathbb{S} = \{T \subseteq 2^{<\omega} : T \text{ is a Sacks tree}\}$ with $\leq := \subseteq$.
$\mathbb{S}_\alpha$: the countable support iteration (c.s.i.) of Sacks forcing of length α.
Sacks model: Forcing with $\mathbb{S}_{\omega_2}$ over a model of CH.

In the Background of this talk: we use a combinatorial notion for the Sacks model due to Miller (1983).

Main result

Theorem (H., Szewczak, Zdomskyy, 2023)

Every Menger totally imperfect subspace of 2^ω has size atmost $\mathfrak{d} = \omega_1 < \mathfrak{c} = \omega_2$ in the Sacks model.

This result may be thought of as a kind of Perfect Set Property for Menger subspaces of 2^ω: Either such a space contains a copy of 2^ω, or has size $< \mathfrak{c}$.

Proof (Sketch).

Choose $\alpha < \omega_2$ s.t. $X \cap V[G_\alpha] \in V[G_\alpha]$, $X \cap V[G_\alpha]$ totally imperfect Menger and if $P, P' \subseteq 2^\omega$ are perfect sets and coded in $V[G_\alpha]$, and $P' \subseteq P \setminus (X \cap V[G_\alpha])$, then $P' \subseteq P \setminus X$. W.l.o.g. $V = V[G_\alpha]$.

Now prove: $X \subseteq V$

Assume not. Let $p_0 \Vdash$ "$\tau \in X \setminus V$". By Sacks combinatorics, we get a condition $p \leq p_0$, a perfect set $K \subseteq (2^\omega)^{\mathrm{supp}(p)}$ and a continuous map $h : K \to 2^\omega$ s.t. $h : K \to h[K]$ is a homeomorphism and $p \Vdash$ "$\tau \in h[K]$" (and all objects are coded in V). By the totally imperfect Menger property of $h^{-1}[X \cap V]$) find perfect $K' \subseteq K \setminus h^{-1}[X \cap V]$.

I.e. $h[K'] \subseteq h[K] \setminus (X \cap V)$. By assumption on V, $h[K'] \subseteq h[K] \setminus X$ holds in $V[G_{\omega_2}]$. Again by Sacks combinatorics find $q \leq p$ such that $q \Vdash$ "$\tau \in h[K']$", hence $q \Vdash$ "$\tau \notin X$". Contradiction. □

Motivation: Tukey order

Theorem (Gartside, Mamatelashvili, 2016)

Let M be a metrizable separable space. Then $\mathcal{K}(M) \not\geq_T \mathcal{K}(\mathbb{Q})$ iff $\mathcal{K}(M)$ is hereditarily Baire.

Theorem (Gartside, Medini, Zdomskyy, 2016)

$\mathcal{K}(M)$ is hereditarily Baire iff $2^\omega \setminus M$ is Menger.

Corollary

If there are exactly $\mathfrak{c}$-many Menger subspaces of 2^ω, then there exists $X \subseteq 2^\omega$ such that $\mathcal{K}(M) \leq_T \mathcal{K}(X)$ for all M with $\mathcal{K}(M) \not\geq_T \mathcal{K}(\mathbb{Q})$.

Question

Is it consistent that there are exactly $\mathfrak{c}$-many Menger subspaces of 2^ω? How about Hurewicz subspaces?

Fact

There are atleast $\mathfrak{c}^{\mathfrak{d}}$-many (resp. $\mathfrak{c}^{\mathfrak{b}}$-many) Menger (resp. Hurewicz) subspaces.

Note that by the main result:
There are exactly $\mathfrak{c}$-many Menger totally imperfect subspaces of 2^ω in the Sacks model.

An essential ingredient in the background of the main result was the Menger game.

Menger Game

It is played on X by two players, I and II, as follows:

For $n \in \omega$, in round n: player I picks an open cover $\mathcal{U}_n$ of X and player II picks a finite $\mathcal{V}_n \subseteq \mathcal{U}_n$.
Player II wins the game if the family $\{\cup \mathcal{V}_n : n \in \omega\}$ covers X, and player I wins otherwise.

The Hurewicz game on X is played in the same way, player II wins the game if the family $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X, and player I wins otherwise.

Theorem (Hurewicz, 1927)

X is Menger if and only if player I has no winning strategy in the Menger game played on X.

Theorem (Scheepers, 1996)

X is Hurewicz if and only if player I has no winning strategy in the Hurewicz game played on X.

In order to use our methods from the proof of the main result to get more and deeper results, we need to make some modifications to the Menger game.

Grouped Menger game

is played on X by players, I and II as follows:

Round 0:

Player I selects a natural number $l_0 > 0$, and then the players play the usual Menger game l_0 many subrounds, thus constructing a partial play
$\langle l_0, \mathcal{U}_0, \mathcal{V}_0, \ldots, \mathcal{U}_{l_0-1}, \mathcal{V}_{l_0-1} \rangle$,
i.e., $\mathcal{V}_i \in [\mathcal{U}_i]^{<\omega}$ for all $i < l_0$.

Round 1:

Player I selects a natural number $l_1 > 0$, and then the players play the usual Menger game another l_1 many subrounds, thus by the end of round 1 constructing a partial play
$\langle l_0, \mathcal{U}_0, \mathcal{V}_0, \ldots, \mathcal{U}_{l_0-1}, \mathcal{V}_{l_0-1}; l_1, \mathcal{U}_{l_0}, \mathcal{V}_{l_0}, \cdots, \mathcal{U}_{l_0+l_1-1}, \mathcal{V}_{l_0+l_1-1} \rangle$
i.e., $\mathcal{V}_i \in [\mathcal{U}_i]^{<\omega}$ for all $i < l_0 + l_1$. (Denote $L_{n+1} := l_0 + \ldots + l_n$, $L_0 := 0$)
Player II **wins the game** if $X = \bigcup_{n\in\omega} \bigcap_{i\in[L_n, L_{n+1})} \cup \mathcal{V}_n$,
and player I wins, otherwise.

A new class between Menger and Hurewicz

Definition

In what follows we denote by $\mathcal{GM}$ the family of all $X \subseteq 2^\omega$ such that player I has no winning strategy in the grouped Menger game on X.

$\mathcal{GM}$ in the Sacks model

Theorem (H., Szewczak, Zdomskyy, 2023)

In the Sacks model, if $X \in \mathcal{GM}$, then both X and $2^\omega \setminus X$ are unions of ω_1-many compact subspaces.

Thus, $|\mathcal{GM}| = \mathfrak{c}^{\omega_1} = \mathfrak{c} = \omega_2$ in this model.

The proof uses the same ideas as the proof of the main result.

Definition

X is *Rothberger* if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle U_n : n \in \omega \rangle$ such that $U_n \in \mathcal{U}_n$ and $\{U_n : n \in \omega\}$ is a cover of X.

Characterization for Rothberger spaces:
For $n \in \omega$, in round n: player I picks an open cover $\mathcal{U}_n$ of X and player II picks $U_n \in \mathcal{U}_n$.
Player II **wins the game** if the family $\{U_n : n \in \omega\}$ covers X, and player I wins otherwise.

X is Rothberger iff player I has no winning strategy.

Consonant spaces

Consonant spaces were introduced by Dolecki, Greco and Lechicki in 1995 and characterized by Jordan in 2020 in the following way:

An open cover $\mathcal{U}$ of X is a *k-cover* if any compact $K \subseteq X$ is contained in some $U \in \mathcal{U}$.
A game $G_1(\mathcal{K}, \mathcal{O})$ played on X by two players as follows:

In round n, player I picks an open k-cover $\mathcal{U}_n$ of X, and player II picks $U_n \in \mathcal{U}_n$.

Player II **wins the game**, if $\bigcup_{n\in\omega} U_n = X$, and player II wins otherwise.

$Y \subseteq 2^\omega$ is *consonant* if and only if player I has no winning strategy in $G_1(\mathcal{K}, \mathcal{O})$ on $X := 2^\omega \setminus Y$.

- If X is consonant, then $2^\omega \setminus X$ is Menger.
- If X is Rothberger, then $2^\omega \setminus X$ is consonant.
- If $2^\omega \setminus Y$ is totally imperfect, then Y consonant iff $2^\omega \setminus Y$ Rothberger.
- All Polish spaces are consonant.

Observation

- *If Y is consonant, then $2^\omega \setminus Y \in \mathcal{GM}$.*
- *If X is Hurewicz, then $X \in \mathcal{GM}$.*

Proof (Sketch for consonant spaces).

We can assume all moves of player I in the Grouped Menger game are increasing covers. For each strategy σ for player I in the grouped Menger game we can define a strategy with moves of the form $\{\bigcap_{i\in[L_n,L_{n+1})} U_i : U_i \in \mathcal{U}_i\}$ in $G_1(\mathcal{K},\mathcal{O})$. Since player I is there not winning, player II can respond s.t. $X = \bigcup_{n\in\omega}\bigcap_{i\in[L_n,L_{n+1})} U_i$. I.e., player I is also not winning in the grouped Menger game following σ. □

Note, hence all Rothberger spaces are in $\mathcal{GM}$.

Corollary

There are exactly $\mathfrak{c}$-many consonant and Hurewicz subspaces in the Sacks model.

Open problems

Question

Is every Menger space $X \subseteq 2^\omega$ (respect. its complement) a union of ω_1-many of its compact subspaces in the Sacks model?

Note: $\mathcal{GM}$ contains no ultrafilters. It is known that $\mathfrak{d} = \mathfrak{c}$ implies the existence of a Menger ultrafilter. So Menger is consistently strictly weaker than $\mathcal{GM}$.

Question

Is it consistent that $\mathcal{GM}$ coincides with the family of all Menger (resp. Hurewicz) subspaces of 2^ω? In other words, is it consistent that for every Menger (resp. Hurewicz) $X \subseteq 2^\omega$ player I has no winning strategy in the grouped Menger game on X?

Thank you for your attention!

Obrigado!