Pseudocompact MAD families under weaker assumptions

Vinicius de Oliveira Rodrigues

University of São Paulo

SUMTOPO 2024 - Coimbra

São Paulo Research Foundation (FAPESP) 2023/17856-7

Ginsburg's question

The Vietoris Hyperspace

- Let X be a T_1 topological space.
- Notation: $\exp(X) = \{C \subseteq X : C \neq \emptyset \text{ is closed}\}$
- Notation: given $U \subseteq X$:
  - $U^+ = \{C \in \exp(X) : C \subseteq U\}$.
  - $U^- = \{C \in \exp(X) : C \cap U \neq \emptyset\}$.
- The Vietoris topology of $\exp(X)$ is the topology generated by the sets above, with U open.
- The Vietoris hyperspace of X is $\exp(X)$ endowed with the Vietoris topology. (Vietoris, 1922)

The Vietoris Hyperspace

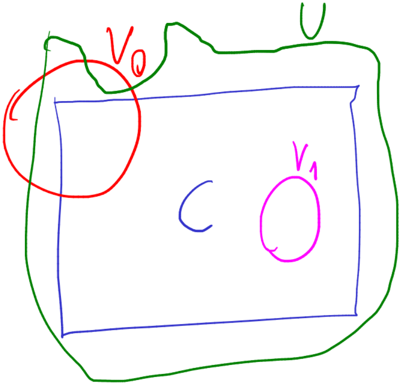


Figure 1: An illustration of $C \in U^+ \cap V_0^- \cap V_1^-$.

The Vietoris Hyperspace - Basic facts

Let X be T_1. Then...

- ...$\exp(X)$ is T_1.
- ...$\exp(X)$ is Hausdorff $\Leftrightarrow$ X is regular.
- ...$\exp(X)$ is Tychonoff $\Leftrightarrow$ $\exp(X)$ is regular $\Leftrightarrow$ X is normal.
- ...$\exp(X)$ is compact $\Leftrightarrow$ X is compact. (Vietoris, 1922)

The Vietoris Hyperspace - Countably compactness

Let X be T_1. Then... (Ginsburg, 1975)

- ...$\exp(X)$ is p-compact $\Leftrightarrow$ X is p-compact (for $p \in \omega^*$).
- ...$\exp(X)$ is countably compact $\Rightarrow \forall n \in \omega\, X^n$ is countably compact.

Corollary

If $\forall \kappa\, X^\kappa$ is countably compact, then $\exp(X)$ is countably compact.

Proof.

$$\forall \kappa\, X^\kappa \text{is countably compact} \Rightarrow \exists p \in \omega^*\, X \text{ is } p\text{-compact}$$

$$\Rightarrow \exists p \in \omega^* \exp(X) \text{ is } p\text{-compact} \Rightarrow \exp(X) \text{ is countably compact.}$$

$\square$

The Vietoris Hyperspace - Pseudocompactness

- A topological space is pseudocompact if $C(X, \mathbb{R}) = C^*(X, \mathbb{R})$.
- A topological space is feebly compact if every sequence of nonempty open sets has a limit point.
- A topological space is p-pseudocompact if every sequence of nonempty open sets has a p-limit point.

Proposition (Ginsburg (1975))

If X is Tychonoff, $\exp(X)$ is pseudocompact $\Leftrightarrow$ $\exp(X)$ is feebly compact.

The Vietoris Hyperspace - Pseudocompactness

Let X be Tychonoff. Then...

- ...$\exp(X)$ is p-pseudocompact $\Leftrightarrow$ X is p-pseudocompact (for $p \in \omega^*$).
- ...$\exp(X)$ is pseudocompact $\Rightarrow$ $\forall n \in \omega\, X^n$ is pseudocompact.

Problem (Ginsburg (1975))

It is natural to ask whether there is any relation between the pseudocompactness (countable compactness) of X^ω and that of $\exp(X)$. It would also be interesting to characterize those spaces X whose hyperspaces are countably compact (pseudocompact).

The Vietoris Hyperspace - Ginsburg's questions

Recall that...

Corollary (Ginsburg (1975))

If X is Hausdorff and $\forall\kappa\, X^\kappa$ is countably compact, then $\exp(X)$ is countably compact.

However...

Proposition (Hrušák, Hernández-Hernández, & Martínez-Ruiz (2007))

There exists X such that $\omega \subseteq X \subseteq \beta\omega$, X^ω is pseudocompact and $\exp(X)$ is not pseudocompact.

Proposition (Ortiz-Castillo, Rodrigues, & Tomita (2018))

There exists X such that $\omega \subseteq X \subseteq \beta\omega$, $\forall\kappa < \mathfrak{h}\ X^\kappa$ is pseudocompact and $\exp(X)$ is not pseudocompact.

Isbell-Mrówka spaces

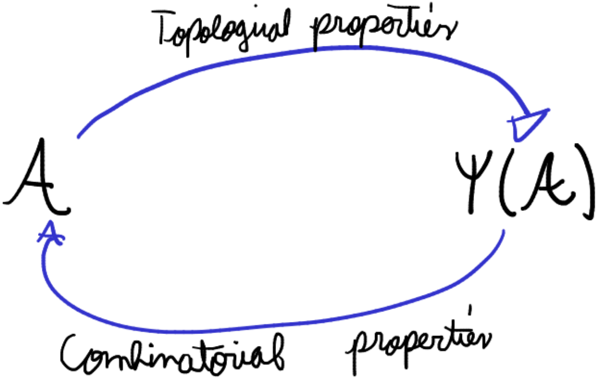
Almost Disjoint Families

- Notation: N is some fixed infinite countable set.
- An almost disjoint family on N is an $\mathcal{A} \subseteq [N]^\omega$ such that:
  - $\mathcal{A}$ is infinite.
  - If $a, b \in \mathcal{A}$ are distinct, $a \cap b$ is finite.
- A MAD family is a maximal almost disjoint family.
- $\mathfrak{a}$ is the least size of a MAD family.
- $\mathfrak{b} \leq \mathfrak{a} \leq \mathfrak{c}$.

Isbell-Mrówka spaces

- Let $\mathcal{A}$ be an almost disjoint family over N.
- We define $\Psi(\mathcal{A}) = N \cup \mathcal{A}$.
- N is open and discrete.
- A basic neighborhood of $a \in A$ consists of the sets $\{a\} \cup \{a \setminus F : F \in [N]^{<\omega}\}$.
- **Basic properties:** $\Psi(\mathcal{A})$ is Tychonoff, locally compact, zero-dimensional, separable and Moore.

Isbell-Mrówka spaces



- e.g. $\mathcal{A}$ is maximal $\Leftrightarrow$ $\Psi(\mathcal{A})$ is pseudocompact $\Leftrightarrow$ $\Psi(\mathcal{A})^\omega$ is pseudocompact.

Pseudocompact MAD families

Isbell-Mrówka spaces and Ginsburg's question

- We say that a MAD family $\mathcal{A}$ is pseudocompact if $\exp(\Psi(\mathcal{A}))$ is pseudocompact.

Is every MAD family pseudocompact?

- According to Hrušák, their ZFC example was inspired by the following:

Proposition (Hrušák, Hernández-Hernández, & Martínez-Ruiz (2007))

If $\mathfrak{p} = \mathfrak{c}$, every MAD family is pseudocompact.

Proposition (Hrušák, Hernández-Hernández, & Martínez-Ruiz (2007))

If $\mathfrak{h} < \mathfrak{c}$, there is a MAD family that is not pseudocompact.

Is every MAD family pseudocompact?

Later, we improved those results as follows:

Proposition (Guzmán, Hrušák, Rodrigues, Todorčević, & Tomita (2022))

Every MAD family is pseudocompact $\Leftrightarrow n(\omega^*) > \mathfrak{c}$ *(*$\Leftrightarrow$ *p-pseudocompact for some* $p \in \omega^*$*, (Corral & Rodrigues, 2024b)).*

*Recall that $n(X)$ is the least cardinality of a collection of open dense sets of X with empty intersection.

Problem (Hrušák, Hernández-Hernández, & Martínez-Ruiz (2007))

Is there a pseudocompact MAD family?

- **Fin-sequences:** A fin sequence (on N) is a sequence $s : \omega \to [N]^{<\omega} \setminus \{\emptyset\}$ of pairwise disjoint sets.

Proposition (Rodrigues & Tomita (2019))

An almost disjoint family is pseudocompact if and only if every fin sequence has an accumulation point in $\exp(\Psi(\mathcal{A}))$.

- **How does such an accumulation point look like?**
- Such accumulation point is neccessarely a subset of $\mathcal{A}$.
- Notation: if $a, X \in [N]^\omega$, $s : \omega \to [\omega]^{<\omega}$, we define $\pi_s(a) = \{n \in \omega : a \cap s(n) \neq \emptyset\}$ and $E_s(X) = \{n \in \omega : s(n) \subseteq X\}$. If $\mathcal{B} \subseteq [N]^\omega$, $\pi_s(\mathcal{B}) = \{\pi_s(a) : a \in \mathcal{B}\}$.
- A nonempty $\mathcal{B} \subseteq \mathcal{A}$ is an accumulation point for s if and only if for every $X \subseteq N$ with $\forall a \in \mathcal{B}\, a \subseteq^* X$, the set $\pi_s(\mathcal{B}) \cup \{E_s(X)\}$ is centered.

Fin-intersecting MAD families

- A nonempty $\mathcal{B} \subseteq \mathcal{A}$ is an accumulation point for s if and only if for every $X \subseteq N$ with $\forall a \in \mathcal{B}\, a \subseteq^* X$, the set $\pi_s(\mathcal{B}) \cup \{E_s(X)\}$ is centered.

Definition (Corral & Rodrigues (2024a))

An almost disjoint family $\mathcal{A}$ is said to be fin-intersecting iff for every fin sequence s, there exists $I \in [\omega]^\omega$ such that $\{\{n \in I : s(n) \cap a \neq \emptyset\} : a \in \mathcal{A}\} \setminus [\omega]^{<\omega}$ is centered.

- Every fin-intersecting MAD family is pseudocompact.
- In that case, $\{a \in \mathcal{A} : |\{n \in I : s(n) \cap A \neq \emptyset\}| = \omega\}$ is an accumulation point for s.
- Fin-intersectingness does not imply madness (e.g. it is hereditary).
- There are non fin-intersecting MAD families in ZFC.
- Every almost disjoint family of size $< \mathfrak{s}$ is fin-intersecting.

Fin-intersecting MAD families

Table of known existence of these objects:

Assumption	Best known result	Reference
$\mathfrak{p} = \mathfrak{c}$	Fin-intersecting MAD	Corral & Rodrigues (2024a)
$n(\omega^*) > \mathfrak{c}$	Pseudocompact	Guzmán et al. (2022)
$\exists \mathcal{U} \in \omega^* \, \mathfrak{p}(\mathcal{U}) = \mathfrak{c}$	Pseudocompact	Guzmán et al. (2022)
$\exists \mathcal{U} \in \omega^* \, \mathfrak{p}(\mathcal{U}) > \mathfrak{a}$	Pseudocompact	Guzmán et al. (2022)
$\mathfrak{a} < \mathfrak{s}$	Fin-intersecting MAD	Corral & Rodrigues (2024a)
$\mathfrak{ap} = \mathfrak{s} = \mathfrak{c}$	Fin-intersecting MAD	Corral & Rodrigues (2024a)
Cohen model	Fin-intersecting MAD	Corral & Rodrigues (2024a)
Random model	Fin-intersecting MAD	Corral & Rodrigues (2024a)
$\diamondsuit(\mathfrak{b})$	Pseudocompact	Corral & Rodrigues (2024b)
$\mathfrak{ap} = \mathfrak{c}$	Pseudocompact	Corral & Rodrigues (2024b)

New results

Construction under $\mathfrak{ap} = \mathfrak{c}$

- Let $\mathcal{A}$ be an almost disjoint family.
- $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ are weakly separated by $X \subseteq N$ iff for every $b \in \mathcal{B}$, $|X \cap b| = \omega$ and for every $c \in \mathcal{C}$, $|X \cap c| < \omega$.
- $\mathfrak{ap}$ is the least size of an almost disjoint family which contains a pair of disjoint subsets which cannot be weakly separated.
- $\mathfrak{p} \leq \mathfrak{ap} \leq \mathfrak{b}$.
- $\mathfrak{ap} < \mathfrak{s}$ and $\mathfrak{ap} > \mathfrak{s}$ are both consistent.

Theorem ($\mathfrak{ap} = \mathfrak{c}$)

There exists a pseudocompact MAD family.

- A nonempty $\mathcal{B} \subseteq \mathcal{A}$ is an accumulation point for a fin-sequence C if and only if for every $X \subseteq N$ with $\forall a \in \mathcal{B}\, a \subseteq^* X$, the set $\pi_C(\mathcal{B}) \cup \{E_s(X)\}$ is centered.
- Let $\mathcal{F}$ be the set of all fin-sequences. Enumerate $[N]^\omega \times \mathcal{F} = \{(X_\alpha, C_\alpha) : \alpha < \mathfrak{c}\}$ with each pair appearing $\mathfrak{c}$ times.
- Recursively, define $a_\alpha, \mathcal{B}_\alpha^C \ (C \in \mathcal{F}, \alpha < \mathfrak{c})$.
- $\mathcal{B}_\alpha^C$ is a partial accumulation point for C. If at step α, X_α almost contains every element of $\mathcal{B}_\alpha^{C_\alpha}$ and witnessess that $\mathcal{B}_\alpha^{C_\alpha}$ is not an accumulation point for C_α, we add a_α to it, with $|a_\alpha \setminus X_\alpha| = \omega$.
- We add this new element to every $\mathcal{B}_\alpha^C$ as long as we do not break the required centeredness of $\pi_C(\mathcal{B}_\alpha^C)$.

- Recursively, define $a_\alpha, \mathcal{B}^C_\alpha, I^C_\alpha$ $(C \in \mathcal{F}, \alpha < \mathfrak{c})$ satisfying:
  1. $a_\alpha \in [\omega]^\omega$,
  2. $a_\beta \cap a_\alpha =^* \emptyset$ for $\beta < \alpha$,
  3. $\mathcal{B}^\alpha_C \subseteq \{a_\beta : \beta < \alpha\}$,
  4. $I^\alpha_C = \{\beta < \alpha : a_\beta \in \mathcal{B}^\alpha_C\}$,
  5. $\mathcal{B}^\beta_C \subseteq \mathcal{B}^\alpha_C$ for every $\beta < \alpha$,
  6. $\mathcal{B}^0_C = \emptyset$,
  7. $\mathcal{B}^\alpha_C = \bigcup_{\beta<\alpha} \mathcal{B}^\beta_C$ if α is limit,
  8. $\{\pi_C(a_\beta) : \beta \in I^\alpha_C\}$ is centered.
  9. If $\{\pi_C(a_\beta) : \beta \in I^\alpha_C\} \cup \{\pi_C(a_\alpha)\}$ is centered then $a_\alpha \in \mathcal{B}^{\alpha+1}_C$, otherwise $\mathcal{B}^{\alpha+1}_C = \mathcal{B}^\alpha_C$.
  10. If $\alpha \geq \omega$ and:
      - for every $a \in \mathcal{B}^\alpha_{C_\alpha}$ we have $a \subseteq^* X_\alpha$,
      - $\{\pi_{C_\alpha}(a_\beta) : \beta \in I^\alpha_{C_\alpha}\} \cup \{E_{C_\alpha}(X_\alpha)\}$ is not centered, and
      - For every $k \geq 0$ and $\beta_0, \dots, \beta_{k-1}$, $\bigcup_{n\in J} C_\alpha(n) \setminus X_\alpha \in \mathcal{I}^+(\{a_\beta : \beta < \alpha\})$, where $J = \bigcap_{i<k} \pi_{C_\alpha}(a_{\beta_i})$.

      Then $\{\pi_{C_\alpha}(a_\beta) : \beta \in I^\alpha_{C_\alpha}\} \cup \{\pi_{C_\alpha}(a_\alpha)\}$ is centered and $a_\alpha \not\subseteq^* X_\alpha$.

10. If $\alpha \geq \omega$ and:
    - for every $a \in \mathcal{B}^{\alpha}_{C_\alpha}$ we have $a \subseteq^* X_\alpha$,
    - $\{\pi_{C_\alpha}(a_\beta) : \beta \in I^{\alpha}_{C_\alpha}\} \cup \{E_{C_\alpha}(X_\alpha)\}$ is not centered, and
    - For every $k \geq 0$ and $\beta_0, \dots, \beta_{k-1}$, $\bigcup_{n \in J} C_\alpha(n) \setminus X_\alpha \in \mathcal{I}^+(\{a_\beta : \beta < \alpha\})$, where $J = \bigcap_{i<k} \pi_{C_\alpha}(a_{\beta_i})$.

    Then $\{\pi_{C_\alpha}(a_\beta) : \beta \in I^{\alpha}_{C_\alpha}\} \cup \{\pi_{C_\alpha}(a_\alpha)\}$ is centered and $a_\alpha \not\subseteq^* X_\alpha$.

- For each such J, we define $a_F \subseteq \bigcup_{n \in J} C_\alpha(n) \setminus X_\alpha$ almost disjoint from $\mathcal{A}_\alpha$ ($\mathfrak{a} = \mathfrak{c}$).
- We use the refinement lemma to get an almost disjoint family $b_F \subseteq a_F$.
- Use $\mathfrak{ap}$ to meet all b_F's and avoid $\mathcal{A}_\alpha$.

- $\diamondsuit(\mathfrak{b})$ is the following statement:
- For every family $(F_\alpha : \alpha < \omega_1)$ of Borel functions $F_\alpha : 2^\alpha \to \omega^\omega$, there exists $g : \omega_1 \to \omega^\omega$ such that for every $t \in 2^\omega_1$, the following set is stationary:

$$\{\alpha < \omega_1 : F_\alpha(t|\alpha) \not\geq^* g(\alpha)\}.$$

Theorem (Construction under $\diamondsuit(\mathfrak{b})$)

There exists a pseudocompact MAD family.

- Let $e_\alpha : \omega \to \alpha$ be a bijection for each infinite α.
- We will define Borel F_α's from $T'_\alpha = ([\omega]^\omega)^\alpha \times 2^\alpha \times ([\omega]^{<\omega})^\omega \times [\omega]^\omega$ into ω^ω.
- In this context, $\diamondsuit(\mathfrak{b})$ guarantees the existence of $g : \omega_1 \to \omega^\omega$ such that for every $((a_\xi : \xi < \omega_1), \sigma, C, X) \in ([\omega]^\omega)^{\omega_1} \times 2^{\omega_1} \times ([\omega]^{<\omega})^\omega \times [\omega]^\omega$, the following set is stationary:

$$\{\alpha < \omega_1 : F((a_\xi : \xi < \alpha), \sigma|\xi, C, X) \not\geq^* g(\alpha)\}.$$

- Notation: if $\mathcal{A} = (a_\xi : \xi < \alpha) \in ([\omega]^\omega)^\alpha$ and $\sigma \in 2^\alpha$, we define $\mathcal{A}^\sigma = \{a_\xi : \sigma(\xi) = 1\}$.
- The set T_α of all $t = (\mathcal{A}_\alpha, \sigma, C, X) \in T'_\alpha$ satisfying the following is Borel, thus we only need to define F_α on this set:
  1. $\mathcal{A}$ is an indexed almost disjoint family,
  2. $C : \omega \to [\omega]^{<\omega}$ is a fin-sequence,
  3. For all $a \in \mathcal{A}^\sigma_\alpha$, $a \subseteq^* X$.
  4. $\pi_C(\mathcal{A}_\alpha)$ is centered.
  5. $\pi_C(\mathcal{A}_\alpha) \cup E_C(\mathcal{A}_\alpha)$ is not centered.

- T_α is the set of all $t = (\mathcal{A}_\alpha, \sigma, C, X) \in T'_\alpha$ satisfying:
  1. $\mathcal{A}$ is an indexed almost disjoint family,
  2. $C : \omega \to [\omega]^{<\omega}$ is a fin-sequence,
  3. For all $a \in \mathcal{A}^\sigma_\alpha$, $a \subseteq^* X$.
  4. $\pi_C(\mathcal{A}_\alpha)$ is centered.
  5. $\pi_C(\mathcal{A}_\alpha) \cup E_C(\mathcal{A}_\alpha)$ is not centered.
- Notation:
  - If t is as above and $n \in \omega$, $G^t_n = \{a_{e_\alpha(k)} : k \leq n, \sigma(e_\alpha(k)) = 1\}$.
  - $I^t_n = \bigcap_{k \leq n} \pi_C(G^t_n)$.
  - $N(t)$ is the first natural such that $I^t_{N(t)} \cap E_C(X)$ is finite.
  - $Y^t_k = \bigcup_{m \in I^t_k} C(m) \setminus X$, which is infinite for k.
- If for some k, $Y^t_k \in I(\mathcal{A}_\alpha)$, let $F(t)$ be constantly 0 (the definition does not matter).
- Otherwise, let $F(t)(k) = \min Y^t_k \setminus \bigcup_{i<k} a_{e_\alpha(i)}$.
- Let g be a $\diamondsuit(\mathfrak{b})$-sequence for F.
- Recursively, let $(a_n : n \in \omega)$ be a partition, and $a_\alpha = \bigcup_{n \in \omega} g(\alpha) \setminus \bigcup_{i<n} a_{e_\alpha(i)}$. WLOG a_α is infinite.

References I

Corral, C., & Rodrigues, V. O. (2024a). Fin-intersecting mad families. *Filomat*, *38*(7), 2563–2578.

Corral, C., & Rodrigues, V. O. (2024b). New examples of mad families with pseudocompact hyperspaces. *Topology and its Applications*, *342*, 108773.

Ginsburg, J. (1975). Some results on the countable compactness and pseudocompactness of hyperspaces. *Canadian Journal of Mathematics*, *27*(6), 1392–1399.

Guzmán, O., Hrušák, M., Rodrigues, V. O., Todorčević, S., & Tomita, A. H. (2022). Maximal almost disjoint families and pseudocompactness of hyperspaces. *Topology and its Applications*, *305*, 107872.

References II

Hrušák, M., Hernández-Hernández, F., & Martínez-Ruiz, I. (2007). Pseudocompactness of hyperspaces. *Topology and its Applications*, *154*(17), 3048–3055.

Ortiz-Castillo, Y. F., Rodrigues, V. O., & Tomita, A. H. (2018). Small cardinals and the pseudocompactness of hyperspaces of subspaces of $\beta\omega$. *Topology and its Applications*, *246*, 9–21.

Rodrigues, V. O., & Tomita, A. H. (2019). Small mad families whose isbell-mrówka space has pseudocompact hyperspace. *Fundamenta Mathematicae*, *247*(1), 99.

Vietoris, L. (1922). Bereiche zweiter ordnung. *Monatshefte für Mathematik*, *32*(1), 258–280.