ALPHA-LIMIT SETS

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Outline

1. Basic definitions
   - Sets of limit points
   - An example

2. Comparison of properties

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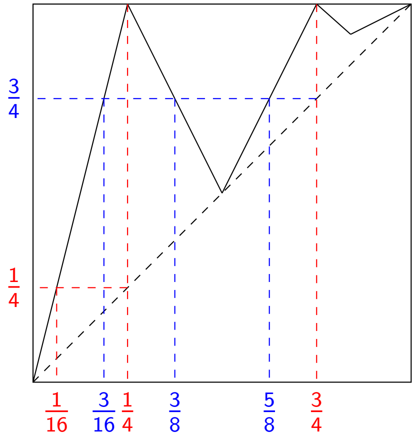
- $B_f(x)$: a set consisting of all backward orbit branches of a point x

Example - difference between sequences



$$f(x) = \begin{cases} 4x, & \text{for } x \in \left[0, \frac{1}{4}\right) \\ -2x + \frac{3}{2}, & \text{for } x \in \left[\frac{1}{4}, \frac{1}{2}\right) \\ 2x - \frac{1}{2}, & \text{for } x \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ \frac{-8}{9}x + \frac{5}{3}, & \text{for } x \in \left[\frac{3}{4}, \frac{21}{25}\right) \\ \frac{1}{2}x + \frac{1}{2}, & \text{for } x \in \left[\frac{21}{25}, 1\right]. \end{cases}$$

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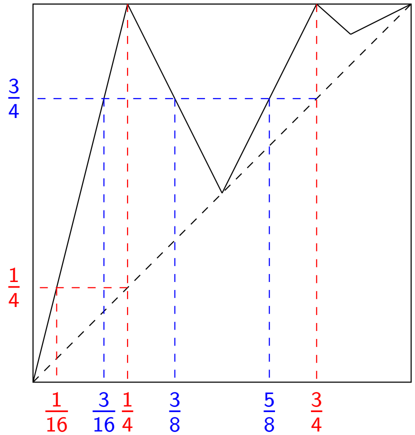

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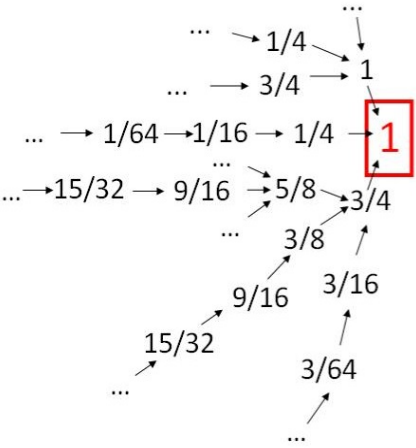

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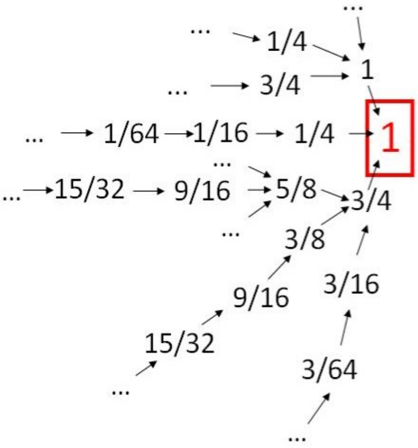

- Preimage sequences: $\left\{1, \frac{3}{4}, \frac{1}{4}, \frac{9}{16}, \frac{15}{32}, \dots\right\}$, $\left\{1, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \dots\right\}$

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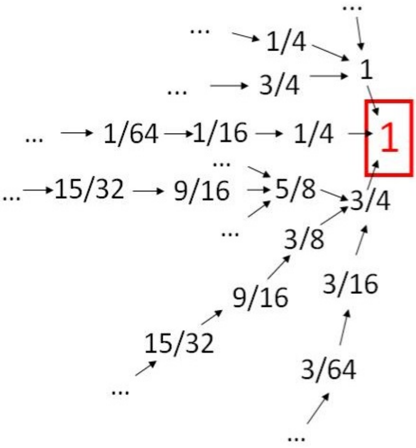

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- An α-limit set of a backward branch $\{x_j\}_{j\leq 0}$: consists of all points y for which there exists a strictly decreasing sequence of negative integers $\{j_i\}_{i\geq 0}$ such that $x_{j_i} \to y$ as $i \to \infty$, denoted by $\alpha_f\left(\{x_j\}_{j\leq 0}\right)$

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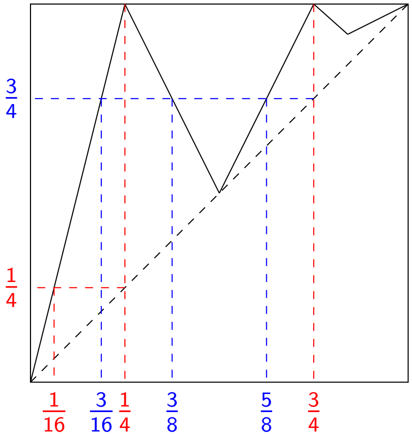

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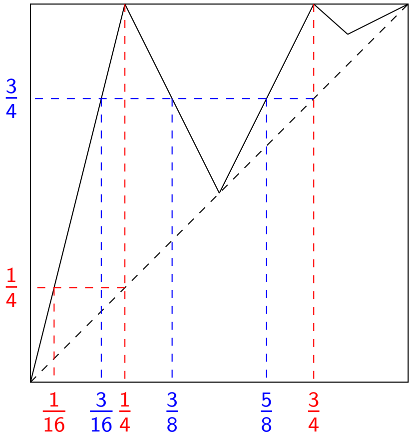

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- $\omega_f(1) = \{1\}$

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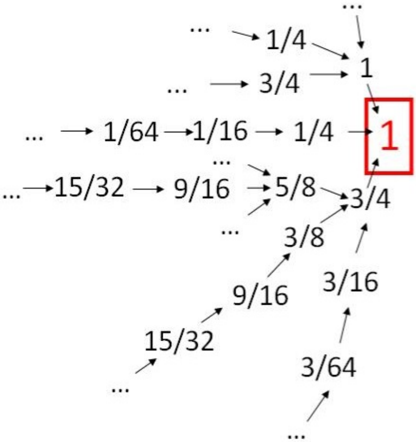

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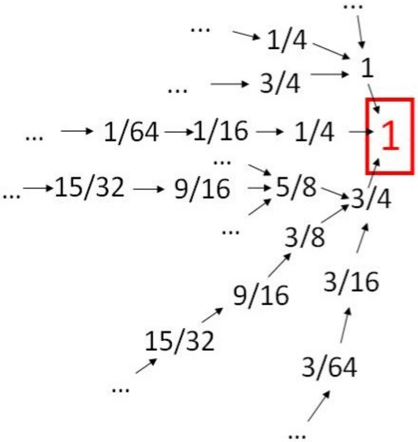

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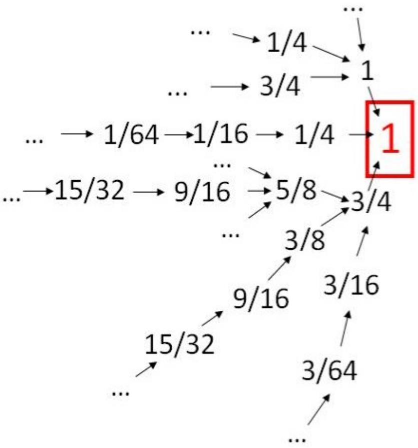

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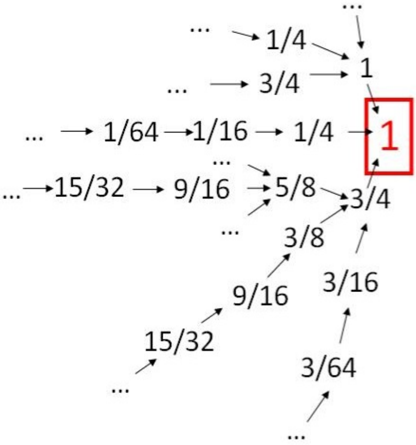

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- $s\alpha(1) = \left\{0, 1, \frac{1}{2}\right\} = \beta(1)$

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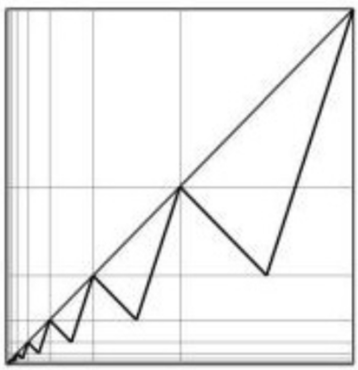
- closeness
  - $\omega(x)$ - always
  - $\alpha(x)$ - always
  - $s\alpha(x)$ - can be open

Comparison of properties of limit sets

- maximal set ($s\alpha(x)$ not contained)[Hantáková and Roth, 2021]



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- Singular set:
  - Semiconjugacy with unit circle and irrational rotation

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- $x \in \beta\,(x)$
- There exists $y \in G$ such that $x \in \beta\,(y)$.

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  - The union of closures is a subset of closure of unions

Thank you for your attention

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