STRONGLY MAZURKIEWICZ MANIFOLDS

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In this note a space means a separable metric space. Cantor n-manifolds (CM(n)) were introduced by Urysohn in 1925 in attempt to describe what a surface is. This class is a generalization of Euclidean manifolds.

Recall that a space X is a Cantor n-manifold (CM(n)) if X cannot be separated by a closed $n-2$ dimensional subset. In other words, X cannot be a union of two proper closed sets whose intersection is of covering dimension $n-2$.

The class CM(n) satisfies the following conditions:

CM-(1) Every connected topological n-dimensional manifold is CM(n).

CM-(2) Every absolute boundary in $\mathbb{R}^{n+1}$ is a CM(n).

CM-(3) Every compact metric space of dimension n contains a CM(n).

In 1957 P. S. Alexandroff introduced the class of continua V^p.

Note that $V^n \in \mathrm{CM(n)}$. Later, N. Hadziivanov (1964) introduced the so-called Strong Cantor n-manifolds (SCM(n)) and has shown that the class of SCM(n) satisfies conditions (1) - (3). Following the line to consider stronger versions of Cantor n-manifolds, in a joint paper with Hadziivanov we introduced Mazurkiewicz n-manifolds (MM(n)).

A space X is MM(n) if for every $(n - 2)$-dimensional subset $M \subset X$ and every two disjoint closed fat (with non empty interiors) sets F and G in X there exists a continuum K such that $K \cap F \neq \emptyset \neq K \cap G$ and $K \cap M = \emptyset$. It is proven in [3] that the class of MM(n) also satisfies conditions (1) - (3).

In the present note we introduce a new class of cantor n-manifolds, Strong Mazurkiewicz n-manifolds (SMM(n)): X ∈SMM(n) if for every subset $M \subset X$ with $dim M \leq n-2$ and for any two points $x; y \in X \setminus M$ there is a continuum K with $x; y \in K$ and $K \cap M = \emptyset$. Note that the class SMM(n) does not satisfy conditions (2) - (3) but obviously
$CM(n) \supsetneq SCM(n) \supsetneq MM(n) \supsetneq SMM(n)$.
Note that $X \in \mathrm{SMM}(n)$ doesn't implies $X \in V^n$.

So far I have not been able to offer a better description of the spaces $\mathrm{SMM}(n)$.

But here's an easy way to construct $\mathrm{SMM}(n)$.

And so, we suppose that:

a) $G \subseteq \mathbb{R}^m$ is a domain in $\mathbb{R}^m$ which means that G is a closure of an open connected set, say $U \subseteq \mathbb{R}^m$.

b) Next let $\{U_k\}_{k=1}^{\infty}$ be a sequence with the following properties:

c) Every U_k is an open region.

d) The set $G_p = G \setminus \bigcup_{k=1}^{p} U_k$ is a closed region for each $p \in \mathbb{N}$.

e) $\lim_{k\to\infty} \operatorname{diam}(U_k) = 0$

f) The space $X \stackrel{\text{def}}{=} G \setminus \bigcup_{k=1}^{\infty} U_k$ is dimensional homogeneous, i.e. $\operatorname{locdim}_x X = n$ for every $x \in X$.

Something about linear connectivity

It is normal to assume that the complement of a zero-dimensional subset of the set with a “big” dimension is linearly connected. As far as I know Vitushkin in his paper “Connection of the variation of a set with metric properties of complements”, Dokl. Akad. Nauk SSSR 114 (1957), 686–689. (Russian) was the first who gave an example of this kind. He constructed (quite a complex) example of a set $V \subset \mathbb{R}^3$ for which $\dim V = 0$ and every two points (x, y, z) and (u, v, w) with $z \neq w$ cannot be joined by an arc which lies outside V.

In the present days we may say more: There exists a zero-dimensional set $M \subset I^{\aleph_0}$ (Hilbert's cube) for which $I^{\aleph_0} / M$ does not contain an arc. To see that it is enough to take a countable base $\{B_i\}$ of $I^{\aleph_0}$ with hereditarily indecomposable boundaries (we may do this according to the paper of R. H. Bing-1951). Then put $M = I^{\aleph_0} \setminus \bigcup_{i=1}^{\infty} \partial(B_i)$. Clearly $\dim M = 0$ and $\bigcup_{i=1}^{\infty} \partial(B_i)$ does not contain an arc (it follows by Sierpinski's theorem).

Thank you!