On uniformly continuous maps between function spaces

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July 1, 2024

Motivation

The C_p-theory was introduced by Arhangel'skii and his students (recall that for a Tychonoff space X the set of all continuous functions on X with the pointwise convergence topology is denoted by $C_p(X)$).

One of the main directions in C_p-theory is the investigation of properties $\mathcal{P}$ such that if $X \in \mathcal{P}$ and $C_p(X)$ is linearly or uniformly homeomorphic to $C_p(Y)$, then $Y \in \mathcal{P}$. Probably, the best results in that direction are Pestov's theorem, stating that if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then $\dim X = \dim Y$, and Uspenskii's theorem that pseudocompactness and compactness are determined by the uniform structure of C_p-spaces.

Let's note that if we consider the function spaces with the uniform convergence, the Pestov's result is not anymore true. Indeed, according to classical Milyutin's theorem if X and Y are uncountable metric compacta, then their function spaces $C(X)$ and $C(Y)$ equipped with the sup metric are linearly homeomorphic.

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Pestov's result was generalized by Gul'ko who proved that $\dim X = \dim Y$ providing $C_p(X)$ and $C_p(Y)$ are uniformly homeomorphic (recall that a map $T : C_p(X) \longrightarrow C_p(Y)$ is uniformly continuous if for every neighborhood U of 0_Y there is a nbd V of 0_X such that $T(f) - T(g) \in U$ provided $f - g \in V$).

Gul'ko's result motivated the investigation of properties $\mathcal{P}$ such that if $X \in \mathcal{P}$ and $C_p(X)$ is uniformly homeomorphic to $C_p(Y)$, then $Y \in \mathcal{P}$.

Another direction in the C_p-theory is to investigate properties $\mathcal{P}$ such that if $X \in \mathcal{P}$ and there is a linear continuous (or uniformly continuous) surjection $T : C_p(X) \longrightarrow C_p(Y)$, then $Y \in \mathcal{P}$. For example, in the class of metrizable spaces completeness is preserved by linear continuous surjections (Baars-de Groot-Pelant), while other absolute Borel classes are preserved by uniformly continuous surjections (Marciszewski-Pelant). Moreover, absolute Borel classes greater than 2 and all projective classes are preserved by homeomorphisms between $C_p(X)$ and $C_p(Y)$ when X, Y are metrizable (Marciszewski).

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On the other hand, Leiderma-Levin-Pestov proved that the Arhangelskii question has a positive answer in dimension 0 when X and Y are metric compacta. The last result was extended for arbitrary compact spaces by Kawamura-Leiderman.

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Main results

Theorem 1 [Eysen-V]

If there is a linear continuous surjection $T : C_p(X) \to C_p(Y)$, then $\dim X = 0$ implies $\dim Y = 0$.

We consider properties $\mathcal{P}$ of metric spaces such that:

(a) if $X \in \mathcal{P}$ and $F \subset X$ is closed, then $F \in \mathcal{P}$;
(b) $\mathcal{P}$ is closed under finite products;
(c) if X is a countable union of closed subsets each having the property $\mathcal{P}$, then $X \in \mathcal{P}$;
(d) if $f : X \to Y$ is a perfect map with countable fibers and $Y \in \mathcal{P}$, then $X \in \mathcal{P}$;
(e) if $X \in \mathcal{P}$ and $F \subset X$, then $F \in \mathcal{P}$.

For example, 0-*dimensionality*, *countable-dimensionality* and *strongly countable-dimensionality* satisfy conditions $(a) - (d)$, while 0-dimensionality and countable-dimensionality satisfy also conditions $(b) - (e)$.

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In the class of metrizable spaces, Theorem 1 has a stronger analogue:

Theorem 2 [Eysen-Leiderman-V]

Let $T : D_p(X) \longrightarrow D_p(Y)$ be a linear continuous surjection and $\mathcal{P}$ be a topological property satisfying either conditions $(a) - (d)$ or $(b) - (e)$. If X is a metric space and Y is perfectly normal, then Y has the property $\mathcal{P}$ provided $X \in \mathcal{P}$.

Here $D_p(X)$ denote either $C_p(X)$ or $C_p^*(X)$, where $C_p^*(X)$ is the set of bounded continuous functions with the pointwise topology.

We say that a surjection $T : D_p(X) \longrightarrow D_p(Y)$ is *inversely bounded* if for every norm bounded sequence $\{g_n\} \subset C^*(Y)$ there is a norm bounded sequence $\{f_n\} \subset C^*(X)$ with $T(f_n) = g_n$ for each n. Note that every linearly continuous surjection $T : C_p^*(X) \longrightarrow C_p^*(Y)$ is inversely bounded.

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Theorem 2 [Eysen-V]

Let $T : D_p(X) \longrightarrow D_p(Y)$ be an inversely bounded uniformly continuous surjection. Then Y is 0-dimensional provided so is X.

Corollary [Eysen-V]

Let $T : C_p(X) \longrightarrow D_p(Y)$ be a linear continuous surjection. Then Y is 0-dimensional provided so is X.

Theorem 2 has a stronger version in case X, Y are metrizable:

Theorem 3 [Eysen-Leiderman-V]

Let $T : D_p(X) \longrightarrow D_p(Y)$ be an inversely bounded uniformly continuous surjection and $\mathcal{P}$ be a topological property satisfying either conditions $(a) - (d)$ or $(b) - (e)$. Then Y has the property $\mathcal{P}$ provided $X \in \mathcal{P}$.

Corollary [Eysen-Leiderman-V]

Let $T : D_p(X) \longrightarrow D_p(Y)$ be an inversely bounded uniformly continuous surjection. If X is either countable-dimensional or strongly countable-dimensional, then so is Y.

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Proofs

When continuous linear surjections are considered, the proofs are based on the ordinary supports of linear functionals.
In the case of uniformly continuous surjections we use the idea of supports introduced by Gul'ko and the extension of that notion introduced by M. Krupski.

THANK YOU