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Dynamics of Induced
Mappings on Symmetric
Products

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- Let X be a metric continuum and  $n \in \mathbb{N}$ .
- Let F<sub>n</sub>(X) be the hyperspace of nonempty subsets of X with at most n points.
- $2^X = \{ A \subseteq X : A \neq \emptyset \text{ is a closed subset of } X \}$
- $F_n(X) = \{A \in 2^X : A \text{ has at most n points}\}$
- If  $1 \le m < n$ , we consider the quotient space  $F_m^n(X) = F_n(X)/F_m(X)$ .

 The Hyperspace 2<sup>x</sup> is considered with the Hausdorff metric.

- Given open subsets  $U_1, \ldots, U_k$  of X, then  $\langle U_1, \ldots, U_k \rangle = \{A \in F_n(X) : A \subset U_1 \cup \ldots \cup U_k \text{ and } A \cap U_i \neq \emptyset \text{ for each } i \in \{1, \ldots, k\}\}$
- The family of sets of the form  $\langle U_1, ..., U_k \rangle$  is a base of the topology in  $F_n(X)$

We denote the quotient mapping by

$$q_m: F_n(X) \longrightarrow F_m^n(X)$$

(or q<sub>m</sub><sup>n</sup>, if necessary)

We denote by  $F_X^m$  the element in  $F_m^n(X)$  such that  $q_m(F_m(X)) = \{F_X^m\}$ .

- Given a mapping f: X → X, the induced mapping 2<sup>f</sup>: 2<sup>X</sup> →2<sup>X</sup> is defined by 2<sup>f</sup>(A) = f(A) (the image of A under f), we consider the induced mappings:
- $f_n: F_n(X) \longrightarrow F_n(X)$ [also denoted  $F_n(f)=2^f \mid_{F_n(X)}$ ]
- $f_m^n$ :  $F_m^n(X) \longrightarrow F_m^n(X)$ .

f<sub>m</sub><sup>n</sup>(X): F<sub>m</sub><sup>n</sup>(X) → F<sub>m</sub><sup>n</sup>(X). 【 is the mapping that makes commutative the following diagram 】

$$F_n(X) \xrightarrow{f_n} F_n(X)$$

$$\downarrow^{q_m} \qquad \qquad \downarrow^{q_m}$$

$$F_m(X) \xrightarrow{f_m^n} F_m(X)$$

## Relations among dynamics of maps

 We study relations among the dynamics of the mappings :

• f : X → X

- $f_n: F_n(X) \longrightarrow F_n(X)$
- $f_m^n$ :  $F_m^n(X) \longrightarrow F_m^n(X)$ .

#### History

- H. Hosokawa 1989 was the first author that studied induced mappings to hyperspaces. This topic has been widely studied. The most common problem being; Given a class of mappings M, determine whether one of the following statements implies another:
- (a) f ∈ *M*
- (b)  $2^f \in M$
- (c)  $f_n \in \mathcal{M}$
- (d)  $f_m^n \in \mathcal{M}$

#### Minimality

Let X be a non-degenerate compact metric space. A mapping f: X → X is *minimal* if there is no nonempty proper closed subset M of X which is invariant under f (invariance of M means that f(M) ⊂ M); equivalently, if the orbit of every point of X is dense in X. The mapping f is totally minimal if f<sup>s</sup> is minimal for each s ∈ N.

Given  $n \in \mathbb{N}$ , we consider the following statements.

- (1) f is minimal,
- (2) f<sub>n</sub> is minimal and
- (3)  $f_1^n$  is minimal.

#### Minimality Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (2)  $\Rightarrow$  (3), (f<sub>n</sub> is minimal  $\Rightarrow$  f<sub>1</sub><sup>n</sup> is minimal)
- (3)  $\Rightarrow$  (1) ( $f_1^n$  is minimal  $\Rightarrow$  f is minimal)
- (2)  $\Rightarrow$  (1) (f<sub>n</sub> is minimal  $\Rightarrow$  f is minimal)

- (1)  $\Rightarrow$  (2) (f is minimal  $\Rightarrow$  f<sub>n</sub> is minimal) and
- (1)  $\Rightarrow$  (3) (f is minimal  $\Rightarrow$   $f_1^n$  is minimal)

#### Question on Minimality

It was asked whether (3) $\Rightarrow$  (2) ( $f_1^n$  is minimal $\Rightarrow$   $f_n$  is minimal). The following theorem solves this question and shows that the question and several results on minimal induced mappings are irrelevant.

# tatements (1)-(4) are equivalent. ot dense orbits on $(F_n(X), f_n)$

 Theorem M Let X be a non-degenerate compact metric space,  $f: X \rightarrow X$  a mapping and  $1 \le m < n$ . Then: (a)  $f_n(F_1(X)) \subset F_1(X)$ , (b)  $f_m^n(F_x^m)=F_x^m$ , (c) for each  $A \in F_m(X)$ , orb $(A, f_n) \subset F_m(X)$ . Thus, orb $(A, f_n) \subset F_m(X)$  $f_n$ ) is not dense in  $F_n(X)$ ,  $f_n$  is not minimal, and (d)  $orb(F_X^m, f_m^n) = \{F_X^m\}$ . Thus,  $orb(F_X^m, f_m^n)$  is not dense in  $F_m^n(X)$  and  $f_m^n$  is not minimal.

#### Proof

- Take a point  $p \in X$ . Then  $f_n(\{p\}) = f(\{p\}) \in F_n(X)$ . Moreover,  $f_m^n(F_X^m) = f_m^n(q_m(\{p\})) = q_m(f_n(\{p\})) =$   $q_m(\{f(p)\}) = F_X^m$ . This proves (a), (b) and (d). The proof of (c) is similar. ■
- Theorem M (b) implies that the mappings  $f_n$  and  $f_m^n$  are never minimal or totally minimal. Then proved results in which minimality or total minimality of  $f_n$  or  $f_m^n$  is either assumed or concluded become irrelevant or partially ir- relevant.

## Irreducibility

Let X be a non-degenerate compact metric space. A mapping f : X → X is *irreducible* if the only closed subset A of X for which f(A) = X is A = X; equivalently, if the orbit of every point of X is dense in X.

Given  $1 \le m < n \in \mathbb{N}$ , we consider the following statements.

- (1) f is irreducible,
- (2) f<sub>n</sub> is irreducible
- (3)  $f_1^n$  is irreducible and
- (4) f<sub>m</sub><sup>n</sup> is irreducible.

#### Irreducibility Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (2)  $\Rightarrow$  (1), (f<sub>n</sub> is irreducible  $\Rightarrow$  f is irreducible)
- (3)  $\Rightarrow$  (1) (f<sub>1</sub><sup>n</sup> is irreducible  $\Rightarrow$  f is irreducible)
- (4)  $\Rightarrow$  (1) (f<sub>m</sub><sup>n</sup> is irreducible  $\Rightarrow$  f is irreducible)

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#### Question on Irreducibility

It is easy to see that the proofs for these results are valid for infinite compact metric spaces without isolated points. The rest of the implications among (1), (2), (3) and (4) were left as questions

## Statements (1)-(4) are equivalent.

- Theorem I.1 Let X be a compact metric space without isolated points,
   f: X → X a mapping and 1 ≤ m < n. If f is irreducible then f<sub>n</sub> is irreducible.
- Theorem I.2 Let X be a compact metric space without isolated points,
   f: X → X a mapping and 1 ≤ m < n. If f<sub>n</sub> is irreducible then f<sub>m</sub><sup>n</sup> is irreducible.
- Corollary I.3 Let X be a compact metric space without isolated points,  $1 \le m < n$  and  $f: X \to X$  a mapping Then the following are equivalent:
  - (a) f is irreducible,
  - (b) f<sub>n</sub> is irreducible,
  - (c) f<sub>m</sub><sup>n</sup> is irreducible

#### **Strong Transitivity**

- Let X be a space. A mapping  $f: X \longrightarrow X$  is **strongly transitive** if for each nonempty open subset U of X, there exists  $r \in \mathbb{N}$  such that  $U_{n=1}^r f^i(U)=X$ .
- Given  $1 \le m < n \in \mathbb{N}$ , we consider the following statements.
- (1) f is strongly transitive,
- (2) f<sub>n</sub> is strongly transitive
- (3)  $f_1^n$  is strongly transitive and
- (4) f<sub>m</sub><sup>n</sup> is strongly transitive.

## Strong Transitivity Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (2)  $\Rightarrow$  (1), (f<sub>n</sub> is str. transitive  $\Rightarrow$  f is str. transitive)
- (2)  $\Rightarrow$  (3), (f<sub>n</sub> is str. transitive  $\Rightarrow$  f<sub>1</sub><sup>n</sup> is str. transitive)
- (2)  $\Rightarrow$  (4), (f<sub>n</sub> is str. transitive  $\Rightarrow$  f<sub>m</sub><sup>n</sup> is str. transitive)
- (1)  $\Rightarrow$  (2) (f is str. transitive  $I \Rightarrow f_n$  is str. transitive)

## Strong Transitivity Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (3)  $\Rightarrow$  (1) ( $f_1^n$  is str. transitive  $\Rightarrow$  f is str. transitive)
- (4)  $\Rightarrow$  (1) (f<sub>m</sub><sup>n</sup> is str. transitive  $\Rightarrow$  f is str. transitive)
- (1)  $\Rightarrow$  (2) (f is str. transitive I  $\Rightarrow$  f<sub>n</sub> is str. transitive)
- (1)  $\Rightarrow$  (3) (f is str. transitive  $\Rightarrow$   $f_1^n$  is str. transitive) and
- (1)  $\Rightarrow$  (4) (f is str. transitive  $\Rightarrow$   $f_m^n$  is str. transitive)

#### Question on Strong Transitivity

It is easy see that the proofs for these results are valid for infinite compact metric spaces without isolated points. The rest of the implications:  $(2) \Rightarrow (3)$ ,  $(2) \Rightarrow (4)$ ,

 $(3)\Rightarrow (4)$  and  $(4)\Rightarrow (3)$  were left as questions.

$$(3),(4) \Rightarrow (2) \text{ and } (3) \iff (4).$$

• Theorem ST.1 Let X be a compact metric space without isolated points,  $f: X \rightarrow X$  a mapping and  $1 \le m < n$ . If  $f_m{}^n$  is strongly transitive then  $f_n$  is strongly transitive.

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#### Turbulence

- Let X be a space. A mapping  $f: X \to X$  is **turbulent** if there are compact non degenerate subsets K and L of X, such that  $K \cap L$  has at most one point and  $K \cup L \subset f(K) \cap f(L)$ .
- Given  $1 \le m < n \in \mathbb{N}$ , we consider the following statements.
- (1) f is turbulent,
- (2)  $f_n$  is turbulent
- (3)  $f_1^n$  is turbulent and
- (4)  $f_m^n$  is turbulent.

#### Turbulence Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (1)  $\Rightarrow$  (2), (f is turbulent  $\Rightarrow$  f<sub>n</sub> is turbulent)
- (3)  $\Rightarrow$  (4), (f<sub>1</sub><sup>n</sup> is turbulent  $\Rightarrow$  f<sub>m</sub><sup>n</sup> is turbulent)

#### Question on Turbulence

The rest of the implications were left as questions, when X is a continuum.

(2),  $(3) \Rightarrow (1)$ , when X is a compact metric space

- Problem T.1 Does one of the statements (2),
   (3) or (4) implies another for a compact metric space?
- Example T.2 There exists a non-degenerate compact metric space X and a mapping f : X → X such that f<sub>2</sub> If f<sub>1</sub><sup>2</sup> are turbulent, but f is not turbulent

X=40704=:neIN3

X = 40? U  $4a_{ming} N?$  U  $4a_{ming}$ 

X = 40?  $U \leq \frac{1}{3m-2}$ : me IN?  $U \leq \frac{1}{3m-4}$ : me IN?  $U \leq \frac{1}{3m}$ : me IN?

 $X = 40? U \frac{1}{3m-2} = me IN? U \frac{1}{3m-4} = me IN? U \frac{1}{3m} = me IN?$   $X = 40? U \frac{1}{3m-2} = me IN? U \frac{1}{3m} = me IN? U \frac{1}{3m} = me IN?$   $X = 40? U \frac{1}{3m-2} = me IN? U \frac{1}{3m} = me IN? U \frac{1}$ 

$$f(x) = \begin{cases} 0; & \text{if } x=0 \\ \text{Ck: } & \text{if } x=azk-1 \end{cases}$$





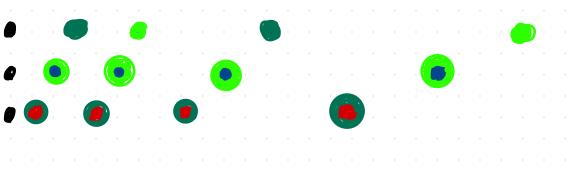






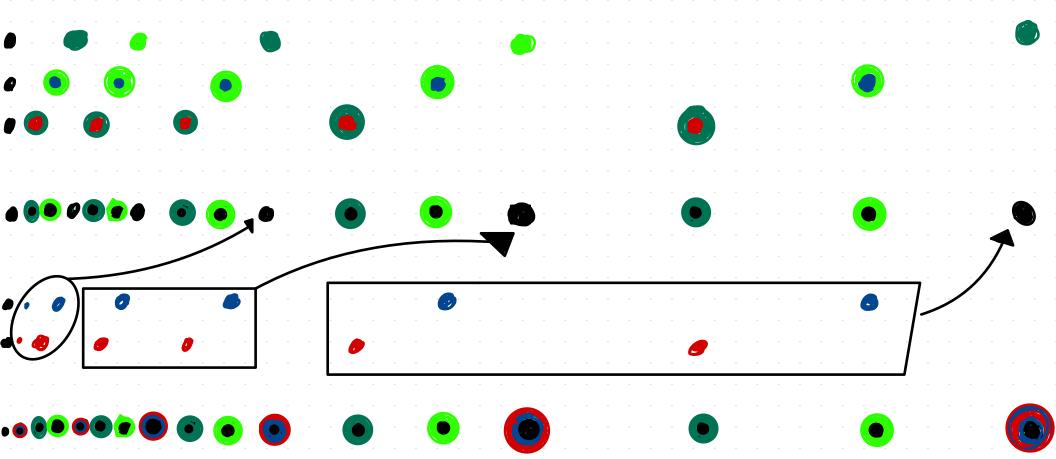


$$f(x) = \begin{cases} 0; & \text{if } x=0 \\ \text{Ck: } \text{if } x = 0 \text{zk}-1 \\ \text{6k: } \text{if } x = 0 \text{zk} \end{cases}$$



$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ C_K; & \text{if } x = \alpha_{2K-1} \\ 6_K; & \text{if } x = \alpha_{2K} \\ \alpha_K; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}$$

f is an onto map.



$$f(x) = \begin{cases} 0; & \text{if } x = 0\\ Cx; & \text{if } x = \alpha_{2}x - 1\\ 6x; & \text{if } x = \alpha_{2}x\\ \alpha x; & \text{if } x \in \{b_{2}x, b_{2}x - 1, C_{2}x, C_{2}x - 1\} \end{cases}$$

$$ax; & \text{if } x \in \{b_{2}x, b_{2}x - 1, C_{2}x, C_{2}x - 1\}$$

 $f(x) = \begin{cases} 0; & \text{if } x = 0 \\ Ck; & \text{if } x = \alpha_{2k-1} \\ 6k; & \text{if } x = \alpha_{2k} \\ \alpha_{k}; & \text{if } x \in \{b_{2k}, b_{2k-1}, C_{2k}, C_{2k-1}\} \end{cases}$   $(ak); & \text{if } x \in \{b_{2k}, b_{2k-1}, C_{2k}, C_{2k-1}\}$ 

Suppose f is turbulent

=> => = K, L compact non obgenerate such that KNL how
at most one point and KUL CF(K) Nf(L)

 $f(x) = \begin{cases} 0; & \text{if } x = 0 \\ Cx; & \text{if } x = a_{2K-1} \\ 6x; & \text{if } x = a_{2K} \end{cases}$   $6x; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\}$   $ax; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\}$ 

Suppose f is turbulent

=> => = K, L compact non degenerate such that KNL how
at most one point and KUL CF(K) Nf(L)

If K=> => CK & KUL, since f-1(CK) =\ark-1? => ark+1 & KNL

 $f(x) = \begin{cases} 0; & \text{if } x = 0 \\ Cx; & \text{if } x = \alpha_{2K-1} \\ 6x; & \text{if } x = \alpha_{2K} \\ \alpha_{K}; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}$   $= \begin{cases} 0; & \text{if } x = \alpha_{2K-1} \\ 6x; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}$ 

Suppose f is turbulent

Suppose f is turbulent

SIRL compact non obgenerate such that Kn

=> 3 K, L compact non degenerate such that KNL has at most one point and KUL CF(K) Nf(L)

19 KZZ CKEKUL, since f-1(CK) =1azk-17 => Ckk+1 GKAL

Since f-1 (aex-1) = 164k-2, 64k-3, C4x-2, C4x-3 & There is p = 164k-2, 64x-2, C4x-3 & OK s.t f(p) = aex-1

(0) if x=0  $f(x) = \begin{cases} C_K; & \text{if } x = \Omega_{2K-1} \\ b_K; & \text{if } x \in \text{lbzk, bzk-1. Czk, Czk-1} \end{cases}$   $\alpha_K; & \text{if } x \in \text{lbzk, bzk-1. Czk, Czk-1} \end{cases}$ is an onto map Suppose of is turbulent => 3 K, L compact non degenerate such that KNL how at most one point and KUL CF(K) Nf(L) If K=>2 Ck & KUL, since f-1(Ck)=1a2k-17 => Ckx+1 & KAL

Since f-1 (ark-1) = 164K-2, 64K-3, C4K-2, C4K-3 & there is pe 164K-2, 64K-3, C4K-2, C4K-3 & NK s.t f(p) = ark-1

Since f-1(p)=Qù for some i>4K-3>2K-1

=> Qù 6 K 1 2> 1 Qù, Q2K-17 C K 1 L a contradiction!

 $f(x) = \begin{cases} 0; & \text{if } x=0 \\ C_K; & \text{if } x=a_{2K-1} \\ 6_K; & \text{if } x=a_{2K} \end{cases}$   $a_K; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\}$ 15 an onto map Suppose + is turbulent => => => K, L compact non degenerate such that KNL how at most one point and KUL CF(K) Nf(L) If K=2 CK & KUL, since f-1(CK) = azk-17 => azk-1 & KAL Since f-1 (ark-1) = 164K-2, 64K-3, C4K-2. C4K-3 } there is PE164K-2, 64K-3, C4K-2. C4K-3 3 NK S. + f(p) = ack-1 Since f-1(p)=ai for some i>4K-3>2K-1 => Chi 6 Kal => Jai, ark-17 CKAL & a contradiction! Thus (KUL) n tex: K723=0 and similarly (KUL) nibk: K723=0 Therefore KULCLAR: KE IN30 46,30 40,3 U 40,3

(0) if x=0 CK; TP X = 02K-1 is an onto map f (x) = 1 6 k; if x = azk ak; if x & {bik, bik-1. Cik, Cik, Cik, }

Suppose of is turbulent => 3 K, 1 compact non degenerate such that KNL has at most one point and kul cf(k) nf(L) IF KZZ CK & KUL, since f-1(CK) = azk-17 => azk+1 G KAL Since f-1 (ark-1) = 164K-2, 64K-3, C4K-2, C4K-3 & There is P 6 1 64K-2, 64K-3, C4X-2, C4K-3 3 NK S. + f(p) = ack-1

Since f-1(p)=ai for some 2>4K-3>2K-1 => Chi 6 Kal => Jai, ark-19 CKAL & a contradiction! Thus (KUL) n Lek: K723=0 and similarly (KUL) nibk: K723=0 Therefore KULCLak: KE IN30 16,30 40,3 U 40,3 U 40,3 ... using similar arguments use end up with

KULC10? a ontradiction

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f(x) = \begin{cases} 0; & \text{if } x = 0 \\ C_{K}; & \text{if } x = \alpha_{2K-1} \\ 6_{K}; & \text{if } x = \alpha_{2K} \\ \alpha_{K}; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}
is an onto map
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To show for is torbulent lef

16 = 4 4 cm, bm? & Fock): m& IN 30 4603?

2 = 4 4 cm, cm? & Fock): m& IN 30 4603?

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f(x) = \begin{cases} 0; & \text{if } x=0 \\ C_K; & \text{if } x=\alpha_{2K-1} \\ 6_K; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}
\alpha_K; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\}
                                                                                                                                                                  is an onto map
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To show fr is torbulent lef

16=4 4 cm, bm ? & F2(X): m& IN ? 0 4 603? 2=4 4 cm, cm? & F2(X): m& 1N 30 4 6033

Then K and L are non degenerate compact subsets of 9200

and Knh = 4503?

$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ C_{K}; & \text{if } x = \alpha_{2K-1} \\ 6_{K}; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}$$

$$(a_{K}); & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\}$$

To show for is torbulent lef

16 = 4 4 clem, but 6 Fock): me IN 30 4 1033

2 = 4 4 clem, cm2 6 Fock): me IN 30 4 1033

Then Kand L are non degenerate compact subsets of 92001
and KAL = 4903?

Given me W, {am, bn? = ff(com), f(am)? = fr (1com, aen?) Efr (1)

$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ C_{K}; & \text{if } x = \alpha_{2K-1} \\ 6_{K}; & \text{if } x = \alpha_{2K} \\ \alpha_{K}; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}$$

$$(a_{K}) = \begin{cases} 0; & \text{if } x = 0 \\ 6_{K}; & \text{if } x \in \{b_{2K}, b_{2K-1}, C_{2K}, C_{2K-1}\} \end{cases}$$

To show for is torbulent let

16 = 4 4 cm, bm? & Fo(X): m& IN ? 0 4 603?

2 = 4 4 cm, cm? & Fo(X): m& IN ? 0 4 603?

Then Ki and L are non degenerate compact subsets of 92001
and Kill = 4903?

Given me W, 1 am, bm? = 1f( [cm), f (am)? = fr (1 cm, am)

E fr (1)

Moreover 1 am, bm? = 1f( bzm), f (am)? = fr (1 bzm, am)

E fr (2)

(0) if x=0  $f(x) = \begin{cases} C_K; & \text{if } x = \Omega_{2K-1} \\ b_K; & \text{if } x \in \text{lbzk, bzk-1. Czk, Czk, } \end{cases}$   $\alpha_K; & \text{if } x \in \text{lbzk, bzk-1. Czk, Czk, } \zeta_{2K-1}$ is an onto map To show for is torbulent lef 76=1 4 clam, bun ? 6 F2 (X): m & IN ? 0 1 103? 2 = 4 1 Clan, Cm? 6 F2(X): m& IN 30 1603? Then K and I are non degenerate compact subsets of Fect and Kn2 = 4903? Given me W, {am, bn? = {f(Czm), f(Qzn)? = fz ({czm, azn?) Moreover (am, bm? = 1f(bzm), f(azm)? = fz (1 bzm, azm?) Since 407 = 4f (0)? = f (440??) = fz (40?) \in fz (X) We have shown that KcfcK) nf(X).

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(0) it f. X=0
  f(x) = \begin{cases} C_K; & \text{if } x = \Omega_{2K-1} \\ 6_K; & \text{if } x = \Omega_{2K} \end{cases}
                                      is an onto map
         ak; if x & {bzk, bzk-1. Czk, Czk-13
 To show fe is torbulent lef
     16= 4 4 cm, bm ? & F2(X): m& 1N ? 0 4 103?
      2=4 4 Clan, Cm? 6 F2(X): me IN 30 11033
 Then K and L are non degenerate compact subsets of Fe CX
 and Knh = 4503?
 Given me W, {am, bm? = ff(Czm), f(Qzn)? = fz ({czm, azn?)
Moreover {am, bn? = {f(62m), f(92m)} = fz ({ b2m, a2m})
                                     e fr (K)
Since 407 = 4+ (0)? = + (4407?) = fr (407) = fr (K) (fr (L)
We have shown that KcfcK) nf(X).
 Similarly 2 Cfr (K) nfr (L). Therefore fr is
                                                                  torbulent
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(0) if x=0  $f(x) = \begin{cases} C_K; & \text{if } x = \alpha_{2K-1} \\ 6_K; & \text{if } x = \alpha_{2K} \end{cases}$ is an onto map ak; if x & {bzk, bzk-1. Czk, Czk-13 To show for is torbulent let 16= 4 3 clam, bun ? 6 F2 (X): m & IN ? 0 4 60 ? } 2=4 10m, Cm? 6 72(X): m61N 90 4603} Then K and L are non degenerate compact subsets of TeX and Knh=49033 Given me W, 1 ambn? = 1f( (czm), f (azn)? = fz (1 czm, azn) Moreover (am, bm? = ff(bzm), f(am)? = fz (1 bzm, am) Since 407 = 4f (0)? = f (440??) = f2 (40?) \in f2 (X) We have shown that KcfcK) nf(x). Similarly 2 Ctr(K) nfr(X). Therefore fr is torbulent Using Ko = 9, (K) and 20=9, (Z) we can prove that = fi is turbulent

## •GRACIAS