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Dynamics of Induced Mappings on Symmetric Products

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Symmetric Products

- Let X be a metric continuum and $n \in \mathbb{N}$.

- Let $F_n(X)$ be the hyperspace of nonempty subsets of X with at most n points.

- $2^X = \{ A \subseteq X : A \neq \emptyset \text{ is a closed subset of } X \}$

- $F_n(X) = \{A \in 2^X : A \text{ has at most n points}\}$

- If $1 \leq m < n$, we consider the quotient space $F_m{}^n(X) = F_n(X) / F_m(X)$.

Symmetric Products

- The Hyperspace 2^X is considered with the Hausdorff metric.

- Given open subsets $U_1, \ldots , U_k$ of X, then $\langle U_1, \ldots , U_k \rangle = \{A \in F_n(X) : A \subset U_1 \cup \ldots \cup U_k$ and $A \cap U_i \neq \emptyset$ for each $i \in \{1,\ldots,k\}\}$

- The family of sets of the form $\langle U_1, \ldots , U_k \rangle$ is a base of the topology in $F_n(X)$

Symmetric Products

- We denote the quotient mapping by

  $q_m: F_n(X) \longrightarrow F_m^n(X)$

  (or q_m^n, if necessary)

  We denote by F_X^m the element in $F_m^n(X)$ such that $q_m(F_m(X)) = \{F_X^m\}$.

Symmetric Products

- Given a mapping $f : X \longrightarrow X$, the induced mapping $2^f : 2^X \longrightarrow 2^X$ is defined by $2^f(A) = f(A)$ (the image of A under f), we consider the induced mappings:

- $f_n : F_n(X) \longrightarrow F_n(X)$
  【also denoted $F_n(f)=2^f|_{F_n(X)}$】

- $f_m^n : F_m^n(X) \longrightarrow F_m^n(X)$.

- $f_m^n(X)$: $F_m^n(X) \longrightarrow F_m^n(X)$. 【 is the mapping that makes commutative the following diagram 】
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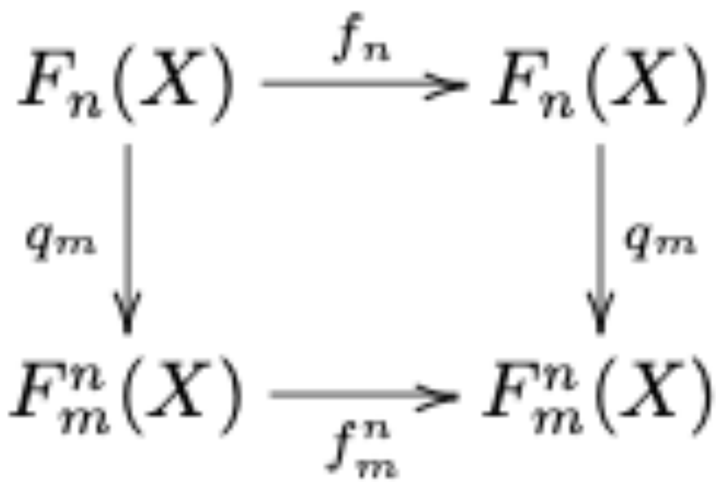

Relations among dynamics of maps

- We study relations among the dynamics of the mappings :

- $f : X \longrightarrow X$

- $f_n : F_n(X) \longrightarrow F_n(X)$

- $f_m{}^n : F_m{}^n(X) \longrightarrow F_m{}^n(X)$.

History

- H. Hosokawa 1989 was the first author that studied induced mappings to hyperspaces. This topic has been widely studied. The most common problem being; Given a class of mappings $\mathscr{M}$, determine whether one of the following statements implies another:
- (a) $f \in \mathscr{M}$
- (b) $2^f \in \mathscr{M}$
- (c) $f_n \in \mathscr{M}$
- (d) $f_m^n \in \mathscr{M}$

Minimality

- Let X be a non-degenerate compact metric space. A mapping $f : X \longrightarrow X$ is ***minimal*** if there is no nonempty proper closed subset M of X which is invariant under f (invariance of M means that $f(M) \subset M$); equivalently, if the orbit of every point of X is dense in X. The mapping f is totally minimal if f^s is minimal for each $s \in \mathbb{N}$.

Given $n \in \mathbb{N}$, we consider the following statements.

(1) f is minimal,

(2) f_n is minimal and

(3) $f_1^{\,n}$ is minimal.

Minimality Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (2) $\Rightarrow$ (3), (f_n is minimal $\Rightarrow$ f_1^n is minimal)
- (3) $\Rightarrow$ (1) (f_1^n is minimal $\Rightarrow$ f is minimal)
- (2) $\Rightarrow$ (1) (f_n is minimal $\Rightarrow$ f is minimal)
- (1) $\not\Rightarrow$ (2) (f is minimal $\not\Rightarrow$ f_n is minimal) and
- (1) $\not\Rightarrow$ (3) (f is minimal $\not\Rightarrow$ f_1^n is minimal)

Question on Minimality

It was asked whether (3)$\Rightarrow$ (2) (f_1^n is minimal$\Rightarrow$ f_n is minimal). The following theorem solves this question and shows that the question and several results on minimal induced mappings are irrelevant.

tatements (1)-(4) are equivalent.
ot dense orbits on $(F_n(X),f_n)$

- **Theorem M** Let X be a non-degenerate compact metric space, $f : X \to X$ a mapping and $1 \leq m < n$. Then:
(a) $f_n(F_1(X)) \subset F_1(X)$,

  (b) $f_m{}^n(F_X{}^m)=F_X{}^m$,
(c) for each $A \in F_m(X)$, $orb(A, f_n) \subset F_m(X)$. Thus, $orb(A, f_n)$ is not dense in $F_n(X)$, f_n is not minimal, and

  (d) $orb(F_X{}^m, f_m{}^n) = \{F_X{}^m\}$. Thus, $orb(F_X{}^m, f_m{}^n)$ is not dense in $F_m{}^n(X)$ and $f_m{}^n$ is not minimal.

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Proof

- Take a point $p \in X$. Then $f_n(\{p\}) = f(\{p\}) \in F_n(X)$. Moreover, $f_m{}^n(F_X{}^m) = f_m{}^n(q_m(\{p\})) = q_m(f_n(\{p\})) = q_m(\{f(p)\}) = F_X{}^m$. This proves (a), (b) and (d). The proof of (c) is similar. ∎
- Theorem M (b) implies that the mappings f_n and $f_m{}^n$ are never minimal or totally minimal. Then proved results in which minimality or total minimality of f_n or $f_m{}^n$ is either assumed or concluded become irrelevant or partially ir- relevant.

Irreducibility

- Let X be a non-degenerate compact metric space. A mapping $f : X \longrightarrow X$ is ***irreducible*** if the only closed subset A of X for which f(A) = X is A = X; equivalently, if the orbit of every point of X is dense in X.

Given $1 \leq m < n \in \mathbb{N}$, we consider the following statements.

(1) f is irreducible,

(2) f_n is irreducible

(3) f_1^n is irreducible and

(4) f_m^n is irreducible.

Irreducibility Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (2) ⇒ (1), (f_n is irreducible ⇒ f is irreducible)
- (3) ⇒ (1) (f_1^n is irreducible ⇒ f is irreducible)
- (4) ⇒ (1) (f_m^n is irreducible ⇒ f is irreducible)
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Question on Irreducibility

It is easy to see that the proofs for these results are valid for infinite compact metric spaces without isolated points. The rest of the implications among (1), (2), (3) and (4) were left as questions

Statements (1)-(4) are equivalent.

- **Theorem I.1** Let X be a compact metric space without isolated points, $f : X \rightarrow X$ a mapping and $1 \leq m < n$. If f is irreducible then f_n is irreducible.
- **Theorem I.2** Let X be a compact metric space without isolated points, $f : X \rightarrow X$ a mapping and $1 \leq m < n$. If f_n is irreducible then f_m^n is irreducible.
- **Corollary I.3** Let X be a compact metric space without isolated points, $1 \leq m < n$ and $f : X \rightarrow X$ a mapping Then the following are equivalent:

  (a) f is irreducible,
  (b) f_n is irreducible,
  (c) f_m^n is irreducible

Strong Transitivity

- Let X be a space. A mapping $f : X \longrightarrow X$ is ***strongly transitive*** if for each nonempty open subset U of X, there exists $r \in \mathbb{N}$ such that $\bigcup_{n=1}^{r} f^{i}(U)=X$.
- Given $1 \leq m < n \in \mathbb{N}$, we consider the following statements.

(1) f is strongly transitive,

(2) f_n is strongly transitive

(3) f_1^n is strongly transitive and

(4) f_m^n is strongly transitive.

Strong Transitivity Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (2) $\Rightarrow$ (1), (f_n is str. transitive $\Rightarrow$ f is str. transitive)
- (2) $\Rightarrow$ (3), (f_n is str. transitive $\Rightarrow$ f_1^n is str. transitive)
- (2) $\Rightarrow$ (4), (f_n is str. transitive $\Rightarrow$ f_m^n is str. transitive)
- (1) $\not\Rightarrow$ (2) (f is str. transitive l $\not\Rightarrow$ f_n is str. transitive)

Strong Transitivity Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (3) $\Rightarrow$ (1) (f_1^n is str. transitive $\Rightarrow$ f is str. transitive)
- (4) $\Rightarrow$ (1) (f_m^n is str. transitive $\Rightarrow$ f is str. transitive)
- (1) $\not\Rightarrow$ (2) (f is str. transitive l $\not\Rightarrow$ f_n is str. transitive)
- (1) $\not\Rightarrow$ (3) (f is str. transitive $\not\Rightarrow$ f_1^n is str. transitive) and
- (1) $\not\Rightarrow$ (4) (f is str. transitive $\not\Rightarrow$ f_m^n is str. transitive)

Question on Strong Transitivity

It is easy see that the proofs for these results are valid for infinite compact metric spaces without isolated points. The rest of the implications: (2)⇒(3), (2)⇒(4),

(3)⇒(4)and (4)⇒(3) were left as questions.

(3),(4)⇒(2) and (3)⟺(4).

- **Theorem ST.1** Let X be a compact metric space without isolated points, $f : X \to X$ a mapping and $1 \leq m < n$. If f_m^n is strongly transitive then f_n is strongly transitive.
-

Turbulence

- Let X be a space. A mapping $f : X \longrightarrow X$ is ***turbulent*** if there are compact non degenerate subsets K and L of X, such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$.
- Given $1 \leq m < n \in \mathbb{N}$, we consider the following statements.

(1) f is turbulent,

(2) f_n is turbulent

(3) f_1^n is turbulent and

(4) f_m^n is turbulent.

Turbulence Known Results.

In (2016, Barragan, Santiago-Santos and Tenorio) showed that for the case that X is a continuum:

- (1) $\Rightarrow$ (2), (f is turbulent $\Rightarrow$ f_n is turbulent)
- (3) $\Rightarrow$ (4), (f_1^n is turbulent $\Rightarrow$ f_m^n is turbulent)

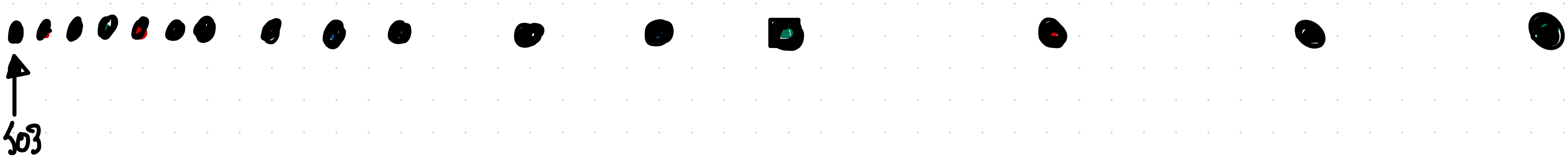
Question on Turbulence

The rest of the implications were left as questions, when X is a continuum.

(2), (3) $\nRightarrow$(1), when X is a compact metric space

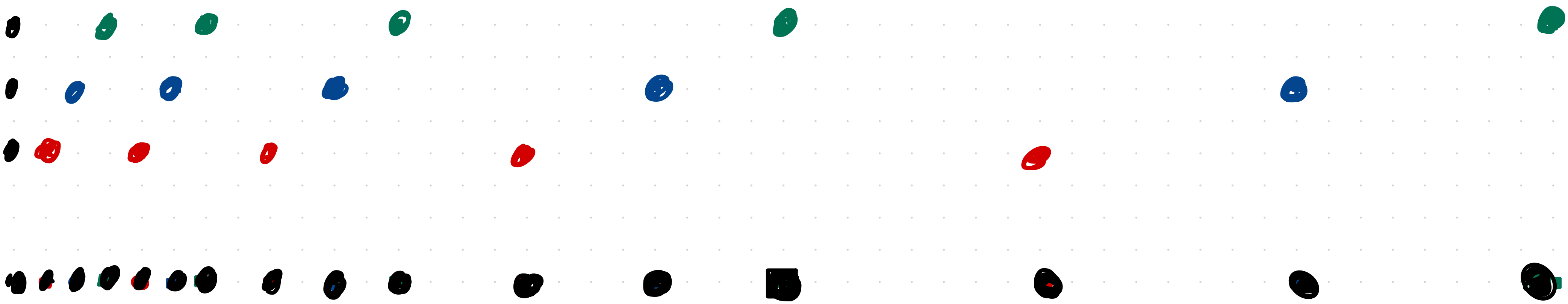
- **Problem T.1** Does one of the statements (2), (3) or (4) implies another for a compact metric space?
- **Example T.2** There exists a non-degenerate compact metric space X and a mapping $f : X \rightarrow X$ such that f_2 If f_1^2 are turbulent, but f is not turbulent

$$X = \{0\} \cup \{\tfrac{1}{n} : n \in \mathbb{N}\}$$



$$X = \{0\} \cup \left\{\frac{1}{3m-2} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3m-1} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3m} : m \in \mathbb{N}\right\}$$

$$X = \{0\} \cup \{a_m : m \in \mathbb{N}\} \cup \{b_m : m \in \mathbb{N}\} \cup \{c_m : m \in \mathbb{N}\}$$



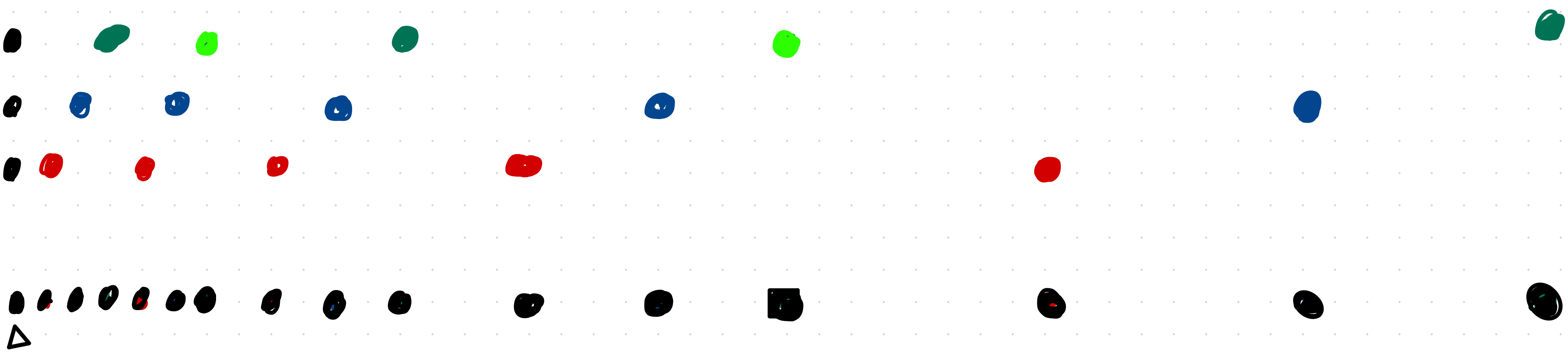
$$X = \{0\} \cup \left\{\frac{1}{3m-2} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3m-1} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3m} : m \in \mathbb{N}\right\}$$

$$X = \{0\} \cup \{a_m : m \in \mathbb{N}\} \cup \{b_m : m \in \mathbb{N}\} \cup \{c_m : m \in \mathbb{N}\}$$

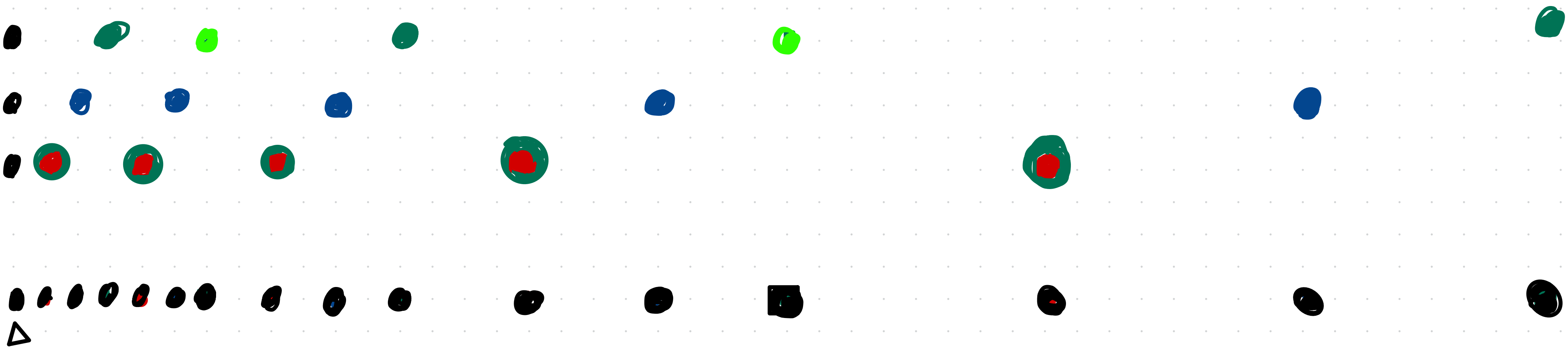
$$X = \{0\} \cup \left\{\frac{1}{3m-2} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3m-1} : m \in \mathbb{N}\right\} \cup \left\{\frac{1}{3m} : m \in \mathbb{N}\right\}$$

$$X = \{0\} \cup \{a_m : m \in \mathbb{N}\} \cup \{b_m : m \in \mathbb{N}\} \cup \{c_m : m \in \mathbb{N}\}$$

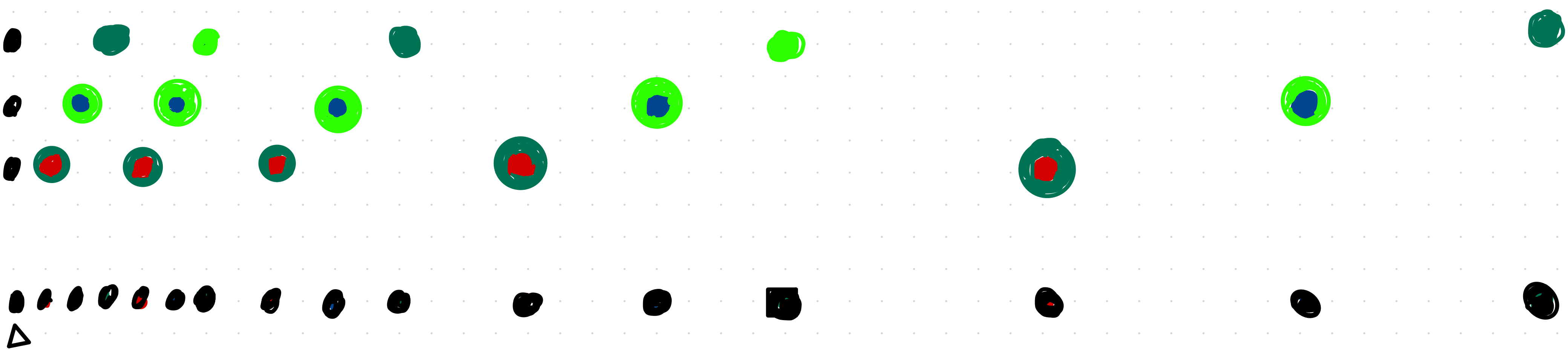
$$X = \{0\} \cup \{a_{2m-1} : m \in \mathbb{N}\} \cup \{a_{2m} : m \in \mathbb{N}\} \cup \{b_m : m \in \mathbb{N}\} \cup \{c_m : m \in \mathbb{N}\}$$



$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k: & \text{if } x = a_{2k-1} \end{cases}$$

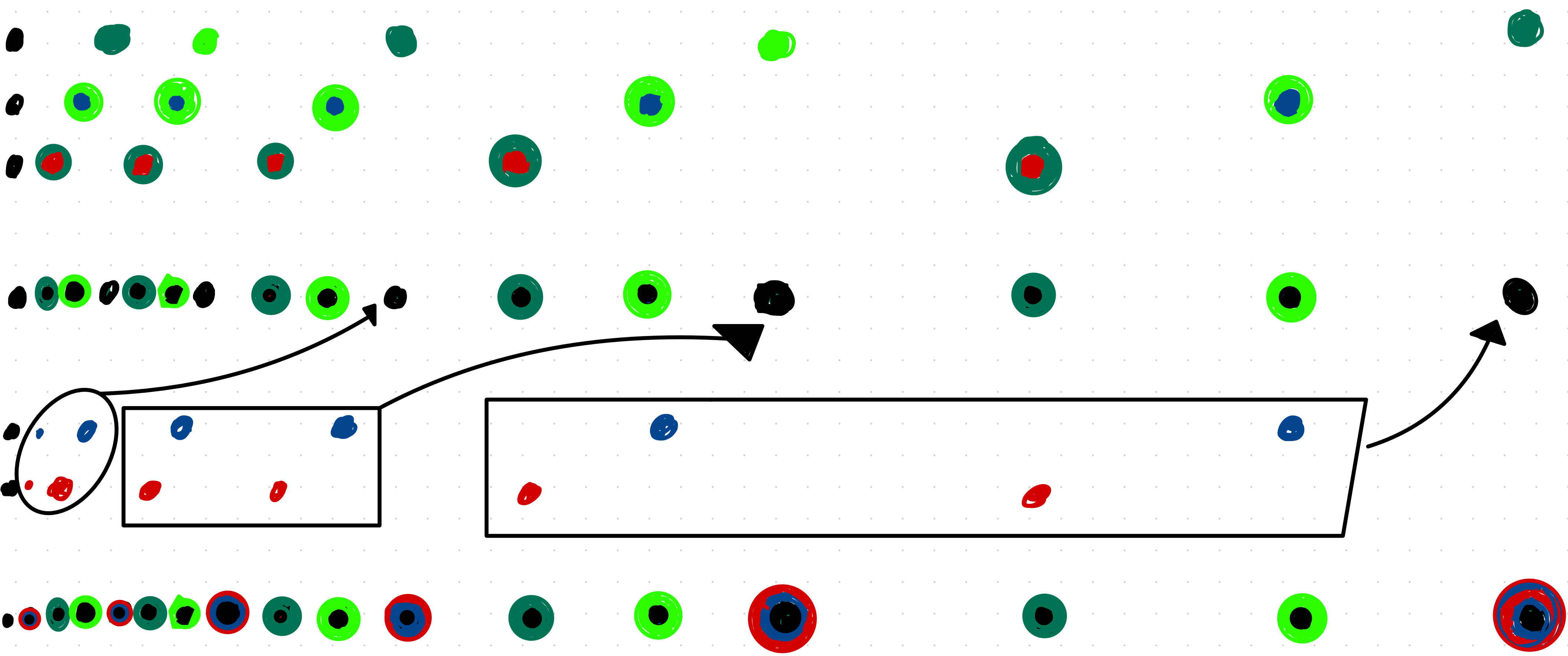


$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k: & \text{if } x = a_{2k-1} \\ b_k: & \text{if } x = a_{2k} \end{cases}$$



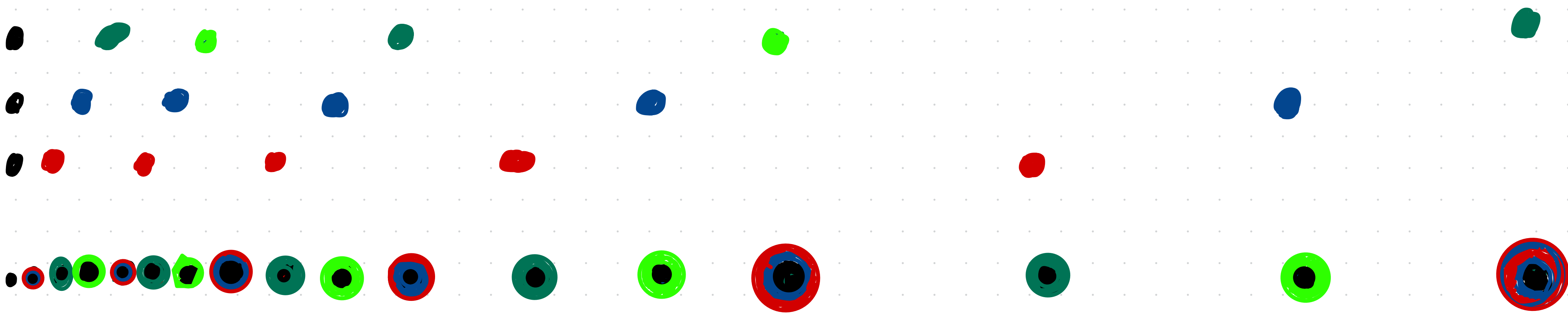
$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

f is an onto map.

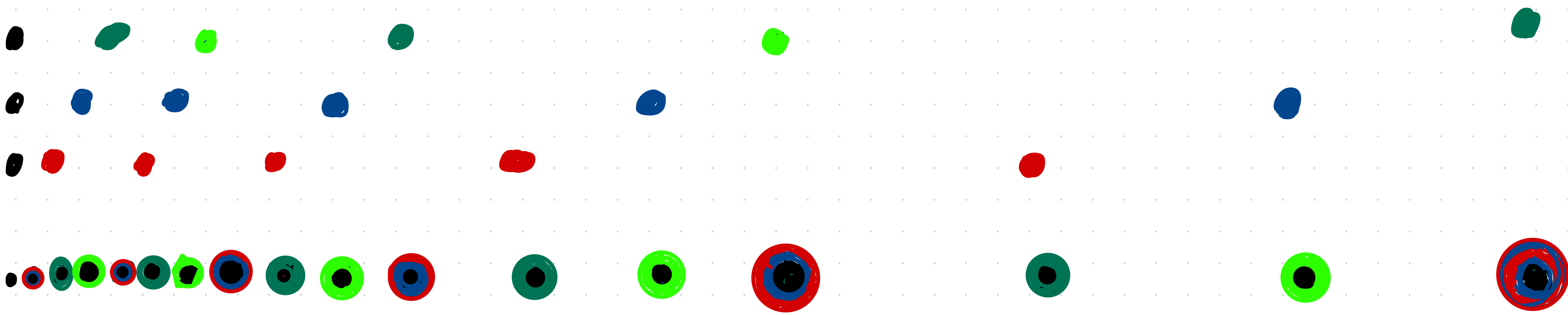


$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$ is an onto map

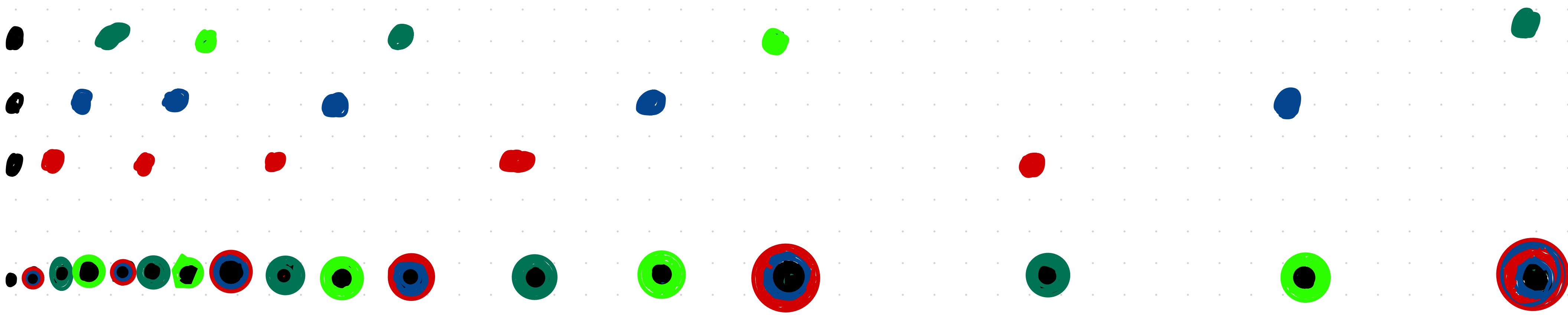


Suppose f is turbulent

$\Rightarrow \exists\, K, L$ compact non degenerate such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$

$$f(x) = \begin{cases} 0; & \text{if } x=0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



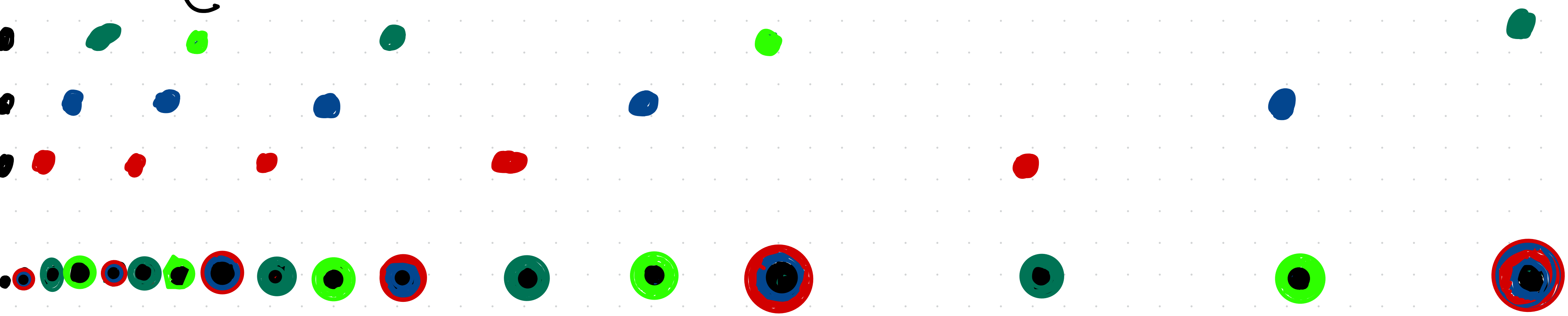
Suppose f is turbulent

$\Rightarrow \exists\, K, L$ compact non degenerate such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$

if $k \geq 2$ $c_k \in K \cup L$, since $f^{-1}(c_k) = \{a_{2k-1}\} \Rightarrow a_{2k+1} \in K \cap L$

$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



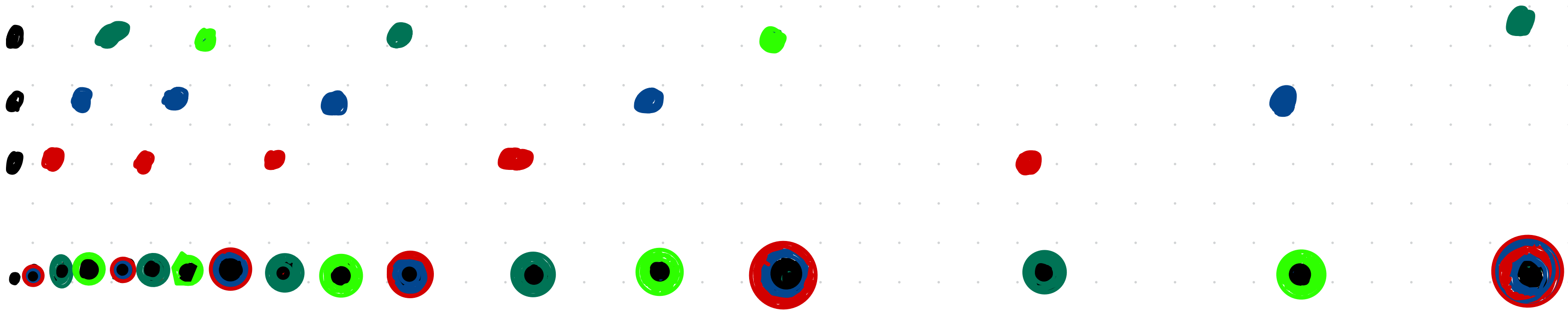
Suppose f is turbolent

$\Rightarrow \exists\, K, L$ compact non degenerate such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$

if $k \geq 2$ $c_k \in K \cup L$, since $f^{-1}(c_k) = \{a_{2k-1}\} \Rightarrow a_{2k+1} \in K \cap L$

Since $f^{-1}(a_{2k-1}) = \{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\}$ there is $p \in \{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\} \cap K$ s.t. $f(p) = a_{2k-1}$

$$f(x) = \begin{cases} 0; & \text{if } x=0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$ is an onto map



Suppose f is turbulent

$\Rightarrow \exists K, L$ compact non degenerate such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$

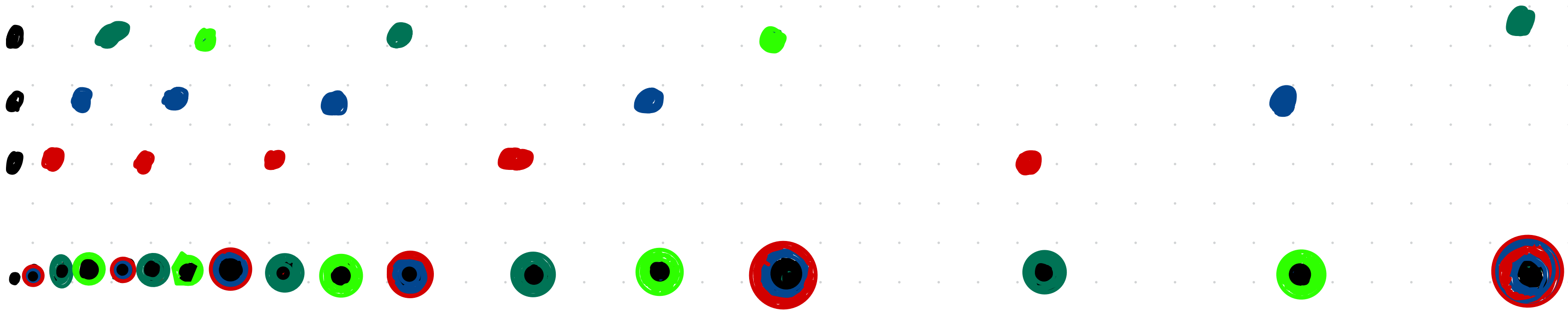
If $k \geq 2$ $\quad c_k \in K \cup L$, since $f^{-1}(c_k) = \{a_{2k-1}\} \Rightarrow a_{2k+1} \in K \cap L$

Since $f^{-1}(a_{2k-1}) = \{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\}$ there is $p \in \{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\} \cap K$ s.t. $f(p) = a_{2k-1}$

Since $f^{-1}(p) = a_i$ for some $i > 4k-3 > 2k-1$

$\Rightarrow a_i \in K \cap L \Rightarrow \{a_i, a_{2k-1}\} \subset K \cap L$ ↯ a contradiction!

$$f(x) = \begin{cases} 0; & \text{if } x=0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$ is an onto map



Suppose f is turbulent

$\Rightarrow \exists\, K, L$ compact non degenerate such that $K \cap L$ has at most one point and $K \cup L \subset f(K) \cap f(L)$

If $k \geq 2$ $\quad c_k \in K \cup L$, since $f^{-1}(c_k) = \{a_{2k-1}\} \Rightarrow a_{2k-1} \in K \cap L$

Since $f^{-1}(a_{2k-1}) = \{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\}$ there is $p \in \{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\} \cap K$ s.t. $f(p) = a_{2k-1}$

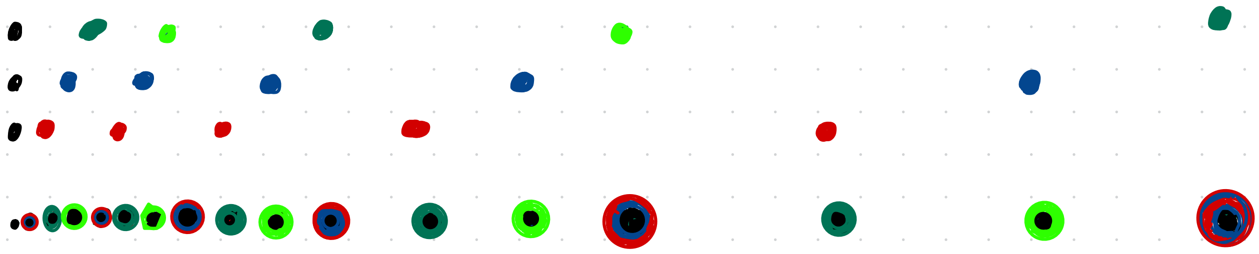
Since $f^{-1}(p) = a_i$ for some $i > 4k-3 > 2k-1$

$\Rightarrow a_i \in K \cap L \Rightarrow \{a_i, a_{2k-1}\} \subset K \cap L$! a contradiction!

Thus $(K \cup L) \cap \{c_k : k \geq 2\} = \emptyset$ and similarly $(K \cup L) \cap \{b_k : k \geq 2\} = \emptyset$

Therefore $K \cup L \subset \{a_k : k \in \mathbb{N}\} \cup \{b_1\} \cup \{c_1\} \cup \{0\}$

$$f(x)=\begin{cases}0; & \text{if } x=0\\ c_k; & \text{if } x=a_{2k-1}\\ b_k; & \text{if } x=a_{2k}\\ a_k; & \text{if } x\in\{b_{2k},b_{2k-1},c_{2k},c_{2k-1}\}\end{cases}$$ is an onto map



Suppose f is turbulent

$\Rightarrow \exists\, K, L$ compact non degenerate such that $K\cap L$ has at most one point and $K\cup L \subset f(K)\cap f(L)$

If $k\geq 2$ $c_k \in K\cup L$, since $f^{-1}(c_k)=\{a_{2k-1}\} \Rightarrow a_{2k-1}\in K\cap L$

Since $f^{-1}(a_{2k-1})=\{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\}$ there is $p\in\{b_{4k-2}, b_{4k-3}, c_{4k-2}, c_{4k-3}\}\cap K$ s.t $f(p)=a_{2k-1}$

Since $f^{-1}(p)=a_i$ for some $i>4k-3>2k-1$

$\Rightarrow a_i\in K\cap L \Rightarrow \{a_i, a_{2k-1}\}\subset K\cap L$ ⧗ a contradiction!

Thus $(K\cup L)\cap\{c_k : k\geq 2\}=\emptyset$ and similarly $(K\cup L)\cap\{b_k : k\geq 2\}=\emptyset$

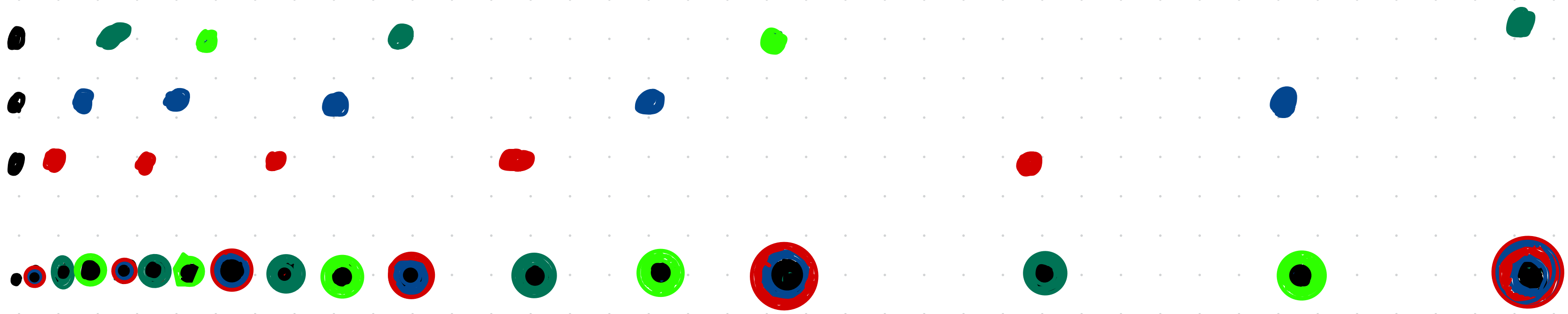
Therefore $K\cup L\subset\{a_k : k\in\mathbb{N}\}\cup\{b_1\}\cup\{c_1\}\cup\{0\}$

... using similar arguments we end up with

$K\cup L\subset\{0\}$ ⧗ a contradiction

$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



To show f_2 is turbulent let

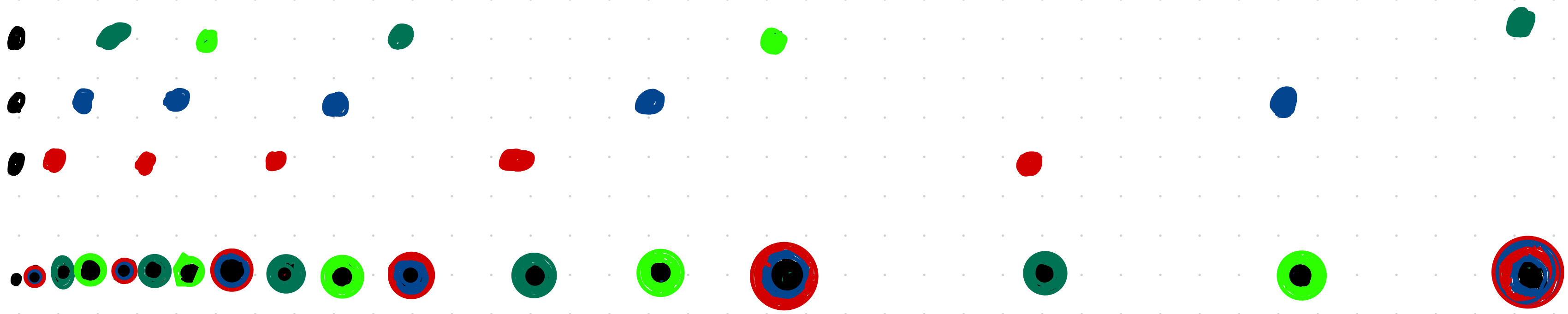
$$K = \{ \{a_m, b_m\} \in F_2(X) : m \in \mathbb{N} \} \cup \{\{0\}\}$$

$$L = \{ \{a_m, c_m\} \in F_2(X) : m \in \mathbb{N} \} \cup \{\{0\}\}$$

=

$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



To show f_2 is turbulent let

$$\mathcal{K} = \{\{a_m, b_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N}\} \cup \{\{0\}\}$$

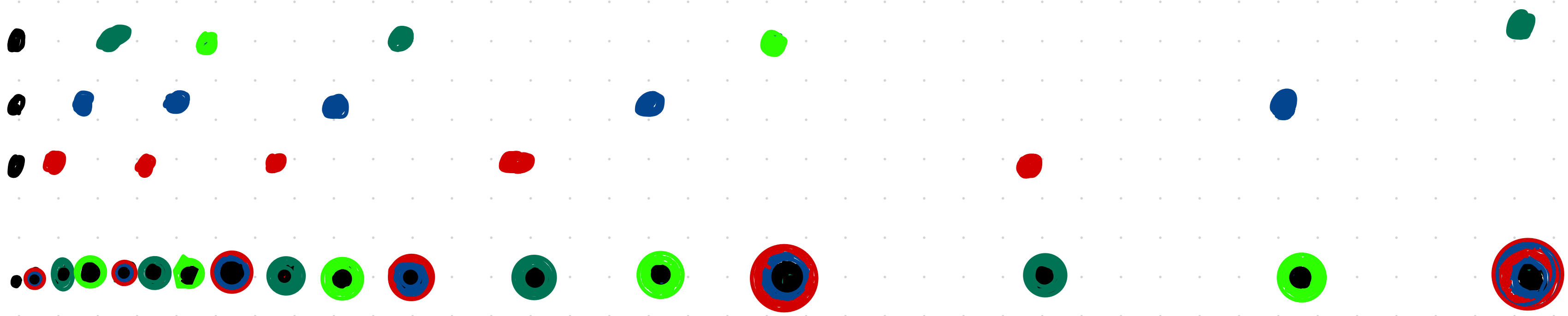
$$\mathcal{L} = \{\{a_m, c_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N}\} \cup \{\{0\}\}$$

Then $\mathcal{K}$ and $\mathcal{L}$ are non degenerate compact subsets of $\mathcal{F}_2(X)$

and $\mathcal{K} \cap \mathcal{L} = \{\{0\}\}$

=

$$f(x)=\begin{cases}0; & \text{if } x=0\\ c_k; & \text{if } x=a_{2k-1}\\ b_k; & \text{if } x=a_{2k}\\ a_k; & \text{if } x\in\{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\}\end{cases}$$ is an onto map



To show f_2 is turbulent let

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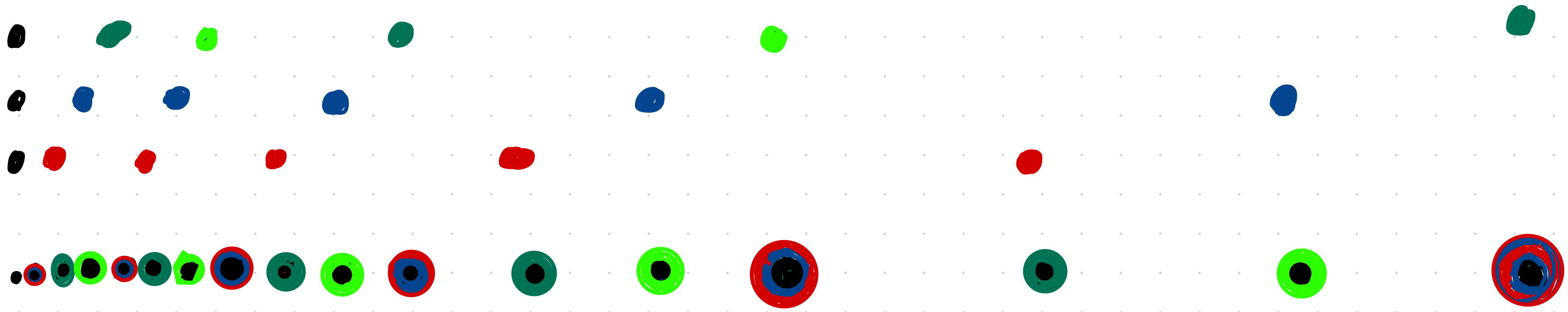
Given $m\in\mathbb{N}$, $\{a_m, b_m\}=\{f(c_{2m}), f(a_{2m})\}=f_2(\{c_{2m}, a_{2m}\})$
$\in f_2(\mathcal{L})$ $=$

$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$ is an onto map



To show f_2 is turbulent let

$$\mathcal{K} = \{ \{a_m, b_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N} \} \cup \{ \{0\} \}$$

$$\mathcal{L} = \{ \{a_m, c_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N} \} \cup \{ \{0\} \}$$

Then $\mathcal{K}$ and $\mathcal{L}$ are non degenerate compact subsets of $\mathcal{F}_2(X)$

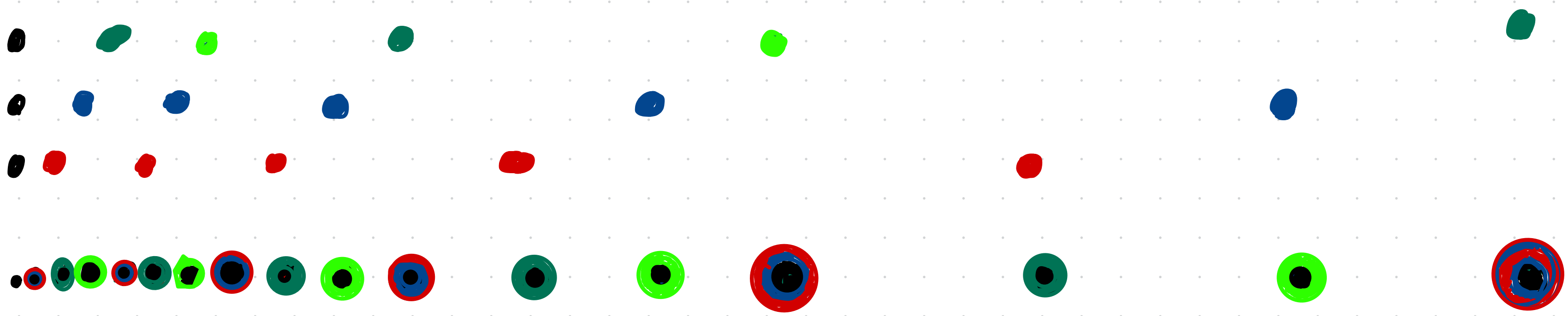
and $\mathcal{K} \cap \mathcal{L} = \{ \{0\} \}$

Given $m \in \mathbb{N}$, $\{a_m, b_m\} = \{f(c_{2m}), f(a_{2m})\} = f_2(\{c_{2m}, a_{2m}\}) \in f_2(\mathcal{L})$ =

Moreover $\{a_m, b_m\} = \{f(b_{2m}), f(a_{2m})\} = f_2(\{b_{2m}, a_{2m}\}) \in f_2(\mathcal{K})$

$$f(x) = \begin{cases} 0 ; & \text{if } x = 0 \\ c_k ; & \text{if } x = a_{2k-1} \\ b_k ; & \text{if } x = a_{2k} \\ a_k ; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



To show f_2 is turbulent let

$$\mathcal{K} = \{ \{a_m, b_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N} \} \cup \{\{0\}\}$$

$$\mathcal{L} = \{ \{a_m, c_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N} \} \cup \{\{0\}\}$$

Then $\mathcal{K}$ and $\mathcal{L}$ are non degenerate compact subsets of $\mathcal{F}_2(X)$ and $\mathcal{K} \cap \mathcal{L} = \{\{0\}\}$

Given $m \in \mathbb{N}$, $\{a_m, b_m\} = \{f(c_{2m}), f(a_{2m})\} = f_2(\{c_{2m}, a_{2m}\}) \in f_2(\mathcal{L})$

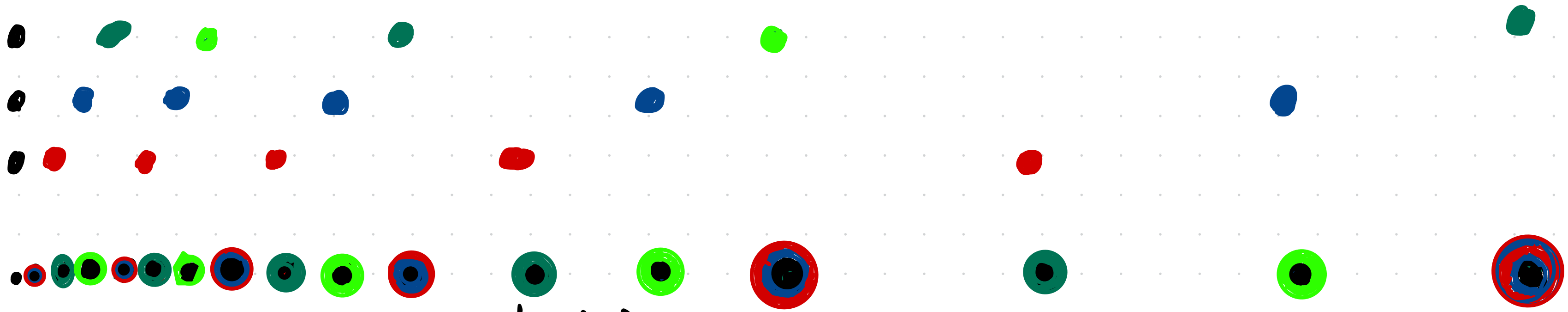
Moreover $\{a_m, b_m\} = \{f(b_{2m}), f(a_{2m})\} = f_2(\{b_{2m}, a_{2m}\}) \in f_2(\mathcal{K})$

Since $\{0\} = \{f(0)\} = f(\{\{0\}\}) = f_2(\{0\}) \in f_2(\mathcal{K}) \cap f_2(\mathcal{L})$

We have shown that $\mathcal{K} \subset f_2(\mathcal{K}) \cap f_2(\mathcal{L})$.

$$f(x) = \begin{cases} 0; & \text{if } x = 0 \\ c_k; & \text{if } x = a_{2k-1} \\ b_k; & \text{if } x = a_{2k} \\ a_k; & \text{if } x \in \{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



To show f_2 is turbulent let

$$\mathcal{K} = \{ \{a_m, b_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N} \} \cup \{\{0\}\}$$

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Then $\mathcal{K}$ and $\mathcal{L}$ are non degenerate compact subsets of $\mathcal{F}_2(X)$

and $\mathcal{K} \cap \mathcal{L} = \{\{0\}\}$

Given $m \in \mathbb{N}$, $\{a_m, b_m\} = \{f(c_{2m}), f(a_{2m})\} = f_2(\{c_{2m}, a_{2m}\}) \in f_2(\mathcal{L})$

Moreover $\{a_m, b_m\} = \{f(b_{2m}), f(a_{2m})\} = f_2(\{b_{2m}, a_{2m}\}) \in f_2(\mathcal{K})$

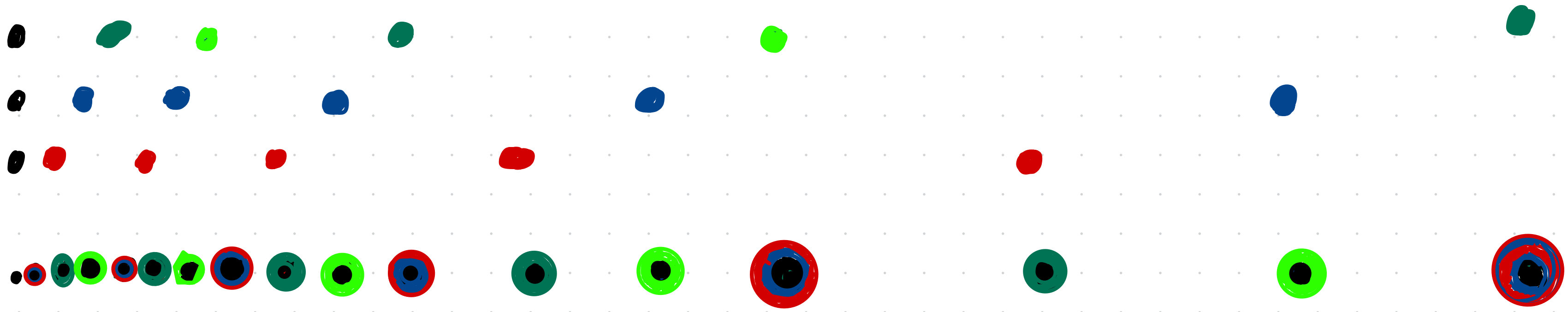
Since $\{0\} = \{f(0)\} = f(\{\{0\}\}) = f_2(\{0\}) \in f_2(\mathcal{K}) \cap f_2(\mathcal{L})$

We have shown that $\mathcal{K} \subset f_2(\mathcal{K}) \cap f_2(\mathcal{L})$.

Similarly $\mathcal{L} \subset f_2(\mathcal{K}) \cap f_2(\mathcal{L})$. Therefore f_2 is turbulent

$$f(x) = \begin{cases} 0; & \text{if } x=0 \\ c_k; & \text{if } x=a_{2k-1} \\ b_k; & \text{if } x=a_{2k} \\ a_k; & \text{if } x\in\{b_{2k}, b_{2k-1}, c_{2k}, c_{2k-1}\} \end{cases}$$

is an onto map



To show f_2 is turbulent let

$\mathcal{K} = \{ \{a_m, b_m\} \in \mathcal{F}_2(X) : m \in \mathbb{N} \} \cup \{\{0\}\}$

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Then $\mathcal{K}$ and $\mathcal{L}$ are non degenerate compact subsets of $\mathcal{F}_2(X)$

and $\mathcal{K} \cap \mathcal{L} = \{\{0\}\}$

Given $m \in \mathbb{N}$, $\{a_m, b_m\} = \{f(c_{2m}), f(a_{2m})\} = f_2(\{c_{2m}, a_{2m}\}) \in f_2(\mathcal{L})$

Moreover $\{a_m, b_m\} = \{f(b_{2m}), f(a_{2m})\} = f_2(\{b_{2m}, a_{2m}\}) \in f_2(\mathcal{K})$

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We have shown that $\mathcal{K} \subset f_2(\mathcal{K}) \cap f_2(\mathcal{L})$.

Similarly $\mathcal{L} \subset f_2(\mathcal{K}) \cap f_2(\mathcal{L})$. Therefore f_2 is turbulent

Using $\mathcal{K}_0 = g_1(\mathcal{K})$ and $\mathcal{L}_0 = g_1(\mathcal{L})$ we can prove that f_1^2 is turbulent ∎ =

- GRACIAS