From Distances and Operator Norms to Normed Categories

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Talk based on joint work with M.M. Clementino (University of Coimbra) and Dirk Hofmann (University of Aveiro)

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What to expect

- Norms making “bad” categories “good”: metric spaces and normed vector spaces
- Quantale-valued norms for sets and categories
- Intermezzo on the Hausdorff metric
- $\mathcal{V}$-enriched categories *versus* $\mathcal{V}$-normed categories
- Cauchy sequences and normed convergence
- Three principal example categories
- The Cauchy cocompletion of a normed category
- Banach’s Fixed Point Theorem

A “good” category of metric spaces?

Classically, a (Fréchet, 1906) metric $d : X \times X \longrightarrow [0, \infty]$ on a set X must satisfy:

Fin	$d(x, y) < \infty$
$\bullet$-Inq	$0 \geq d(x, x)$ (*i.e.* $0 = d(x, x)$)
Sep	$d(x, y) = 0 = d(y, x) \Longrightarrow x = y$
Sym	$d(x, y) = d(y, x)$
Δ-Inq	$d(x, y) + d(y, z) \geq d(x, z)$

Suppose we let these metric spaces be the objects of a category, called

$$\mathsf{Met}_{\mathsf{Fr\acute{e}chet}} ,$$

with morphisms $f : X \to Y$ to satisfy

1-Lip	$d_X(x, x') \geq d_Y(fx, fx')$

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Some shortcomings of $\mathrm{Met}_{\text{Fréchet}}$, Hausdorff's 1914 observations

- $\mathrm{Met}_{\text{Fréchet}}$ has finite limits, but not all countable products (not even of 2-point spaces).
- Any two non-empty spaces fail to admit a coproduct.
- Neither its cartesian structure nor its natural monoidal structure are closed.
- The (non-symmetrized) Hausdorff distance

$$d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$$

for $A, B \subseteq X$ will generally satisfy *only* $\bullet$-Inq and $\triangle$-Inq.

BUT these two conditions hold even when the given distance function on X satisfies none of the other conditions!

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Objects are required to satisfy only •-Inq and Δ-Inq; as before, morphisms satisfy 1-Lip.

Then:

- Met_1 is topological over Set, hence complete and cocomplete.
- Met_1 is symmetric monoidal closed (and, hence, allows for the formation of "good" function spaces, although it is not cartesian closed).

Requiring also the Sym condition would not obstruct these properties,

but it would obstruct *viewing individual objects as small $\mathcal{R}_+$-enriched categories*:

	$\mathcal{R}_+ = ([0,\infty], \geq, +, 0)$	$\mathsf{Ab} = (\mathsf{Ab}, \to, \otimes, \mathbb{Z})$
$0 \geq d(x,x)$	$0 \geq X(x,x)$	$\mathbb{Z} \longrightarrow \mathcal{X}(x,x)$
$d(x,y) + d(y,z) \geq d(x,z)$	$X(x,y) + X(y,z) \geq X(x,z)$	$\mathcal{X}(x,y) \otimes \mathcal{X}(y,z) \longrightarrow \mathcal{X}(x,z)$

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Lawvere 1973: Importing enriched category theory to metric spaces

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Regarding individual spaces as *ordinary* small categories

Every set X gives rise to the (indiscrete) category over X:

$$\mathrm{i}X$$

Its objects are the points of X, and for all $x, y \in X$ there is exactly one morphism $x \to y$:

$$x \xrightarrow{(x,y)} y$$

If X is a (Lawvere) metric space, then this morphism has a "norm":

$$|(x,y)| := X(x,y)$$

With $\mathrm{id}_x = (x,x)$ and the composable morphisms $f := (x,y),\ g := (y,z)$ we have

$$0 \geq |\mathrm{id}_x| \qquad\qquad |f| + |g| \geq |g \cdot f|$$

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Catering to the Working Mathematician: “What about morphisms?”

Met$_\infty$

Again, objects must satisy $\bullet$-Inq and Δ-Inq, but morphisms are now arbitrary mappings.

This seems like an utterly uninteresting category: Met$_\infty \simeq$ Set !

BUT we could try to make it interesting by providing it with some kind of “operator norm”; that is, by considering for every morphism $f : X \to Y$ (with some arithmetic for 0 and ∞)

$$\mathrm{L}(f) := \inf\{\ell \in [0,\infty] \mid \forall x, x' \in X : \ \ell \cdot d(x,x') \geq d(fx, fx')\}$$

Then:

$$1 \geq \mathrm{L}(\mathrm{id}_X) \qquad\qquad \mathrm{L}(g)\cdot\mathrm{L}(f) \geq \mathrm{L}(g\cdot f)$$

Not to leave the realm of $\mathcal{R}_+$, we could put:

$$|f| := \log^\circ \mathrm{L}(f) \qquad (\text{with } \log^\circ \alpha = \max\{0, \log\alpha\})$$

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Adding structure: What about normed vector spaces?

A norm $\|\text{-}\| : X \to [0, \infty]$ on a (real, say) vector space X must satisfy

Fin	$\|x\| < \infty$
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Hom	$\|ax\| = \|a\|\,\|x\| \quad (a \in \mathbb{R})$
Δ-Inq	$\|x\| + \|y\| \geq \|x + y\|$

Such X becomes a one-object category, making every $x \in X$ a morphism $x : * \to *$, with a "norm" satisfying $\bullet$-Inq and Δ-Inq, and to be composed via vector addition.

These are the objects of the category

$$\mathsf{NVec}_1$$

whose morphisms $f : X \to Y$ are linear maps satisfying

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Some strange (co)limits in NVec_1

For a constant $c > 0$, let $\mathbb{R}_c$ be the 1-dimensional vector space $\mathbb{R}$, normed by $|1|_c = c$.

Consider the following sequence in NVec_1, carried by identity maps:

$$\mathbb{R} = \mathbb{R}_1 \longrightarrow \mathbb{R}_{\frac{1}{2}} \longrightarrow \mathbb{R}_{\frac{1}{3}} \longrightarrow \ldots \longrightarrow \mathrm{colim} = 0$$

Wouldn't it be "nicer" to obtain $\mathbb{R}_0$, *i.e.* to allow $c = 0$?

Likewise, consider the following inverse sequence in NVec_1, carried by identity maps:

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Envisioning SNVec_∞

Facit:

NVec_1 is too tight, both at the object and the morphism levels.

Goals:

- Similarly to the transition from $\mathsf{Met}_{\mathsf{Fréchet}}$ to Met_∞, replace NVec_1 by a category of semi-normed vector spaces and all linear maps,
- with a norm based on the classical operator norm, but one that fits into a general categorical framework of normed categories.
- Introduce a notion of (Cauchy) convergence for sequences of operators that is supported by the categorical framework, and that not only
- yields better behaved sequential (co)limits in the examples just given, but also
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Replacing $\mathcal{R}_+ = ([0,\infty], \geq, +, 0)$ by $\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$

In this talk, a *quantale* (always unital and commutative) is given by

- a complete lattice $(\mathcal{V}, \leq)$
- a commutative monoid $(\mathcal{V}, \otimes, \mathrm{k})$
- satisfying the infinite distributive law $\quad (\bigvee_i u_i) \otimes v = \bigvee_i (u_i \otimes v)$

That is:

$\mathcal{V} = (\mathcal{V}, \leq, \otimes, \mathrm{k})$ is a small, thin, skeletal, cocomplete symmetric monoidal-closed category

Internal hom: $\quad u \leq [v, w] \iff u \otimes v \leq w \qquad (\text{-} \otimes v) \dashv [v, \text{-}] : \mathcal{V} \longrightarrow \mathcal{V}$

Key examples: $\quad \mathcal{R}_+$ (Lawvere) $\qquad 2 = (\{\mathrm{true}, \mathrm{false}\}, \Rightarrow, \wedge, \mathrm{true})$ (Boole)

$$[v, w] = \max\{w - v, 0\} = w \mathbin{\hat{-}} v \qquad [v, w] = (v \Rightarrow w)$$

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$\mathrm{Set}/\!/\mathcal{V} := \mathrm{Fam}(\mathcal{V}, \leq)$

$\mathcal{V}$-normed sets and maps:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "|\text{-}|_A"'] & \arrow[d, phantom, "\leq"] & B \arrow[dl, "|\text{-}|_B"] \\ & \mathcal{V} & \end{tikzcd} \qquad |a|_A \leq |fa|_B$$

$\mathrm{Set}/\!/\mathcal{V}$ is

- topological over Set
- locally presentable
- symmetric monoidal closed

$$A \otimes B = A \times B, \quad |(a,b)| = |a| \otimes |b| \qquad \mathrm{E} = \{*\}, \quad |*| = \mathrm{k}$$

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$$A \otimes B = A \times B, \quad |(a,b)| = |a| \otimes |b| \qquad \mathrm{E} = \{*\}, \quad |*| = \mathrm{k}$$

$$[A,B] = \text{Set}(A,B), \quad |\varphi| = \bigwedge_{a \in A} [|a|, |\varphi a|] \quad \text{(i.e. } |\varphi| \text{ is maximal with } |\varphi| \otimes |a| \leq |\varphi a|)$$

$\mathcal{V}$-normed categories and functors:

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "F"] \arrow[dr, "|\text{-}|_{\mathbb{X}}"'] \arrow[rr, phantom, shift right=6ex, "\leq"] & & \mathbb{Y} \arrow[dl, "|\text{-}|_{\mathbb{Y}}"] \\ & (\mathcal{V}, \otimes) & \end{tikzcd}$$

$$|f|_{\mathbb{X}} \leq |Ff|_{\mathbb{Y}}$$

$$\mathrm{E} \to \mathbb{X}(x,x) \qquad \mathbb{X}(x,y) \otimes \mathbb{X}(y,z) \to \mathbb{X}(x,z)$$

$$* \mapsto 1_x \qquad (f,g) \mapsto g \cdot f$$

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Some first examples of ($\mathcal{V}$-)normed categories

$\mathcal{V} = \mathcal{R}_+$:

- Every metric space X becomes a (small) normed category $\mathrm{i}X$ with $|(x,y)| := X(x,y)$.
- The (large) category Met_∞ becomes normed with $|f| := \log^\circ \mathrm{L}(f)$.
- Similarly we will consider the large normed category SNVec_∞.

$\mathcal{V} = 2$:

- A 2-normed category $\mathbb{X}$ comes with a (characteristic) function $|\text{-}| : \mathbb{X} \to 2$, *i.e.* with a subclass $\mathcal{S}$ of morphisms in $\mathbb{X}$, satisfying

$$1_x \in \mathcal{S} \qquad\qquad f \in \mathcal{S} \;\&\; g \in \mathcal{S} \Longrightarrow g \cdot f \in \mathcal{S}$$

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Extrapolating from metric spaces to $\mathcal{V}$-categories: $\mathsf{Met}_1 \rightsquigarrow \mathcal{V}$-$\mathsf{Cat}$

A $\mathcal{V}$-category structure on a set X is a function $X(\text{-},\text{-}) : X \times X \to \mathcal{V}$ satisfying

$$\mathrm{k} \leq X(x,x) \qquad\qquad X(x,y) \otimes X(y,z) \leq X(x,z)$$

A $\mathcal{V}$-functor is a map $f : X \to Y$ of (small) $\mathcal{V}$-categories satisfying

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- $\mathcal{R}_+$-Cat $=$ Met_1
- 2-Cat $=$ Ord
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$\mathcal{V}$-categories *vs.* (Set$//\mathcal{V}$)-categories

$$\begin{tikzcd} \mathcal{V} \arrow[r, shift left, "\mathrm{i}"] \arrow[r, phantom, "\top"] & \mathsf{Set}//\mathcal{V} \arrow[l, shift left, bend left, "\mathrm{s}"] \end{tikzcd} \quad = \mathrm{Fam}(\mathcal{V}, \leq)$$

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Intermezzo on the Hausdorff metric: distributors

$\mathcal{R}_+$-Dist:

- Objects are metric spaces; morphisms $X \overset{\rho}{\longrightarrow\!\!\!\circ\!\!\to} Y$ are "distributors":

$$\begin{tikzcd}[column sep=small] & x \arrow[rr, dash, "\rho(x{,}y)"] & & y \arrow[dr, dash, "Y(y{,}y')"] & \\ x' \arrow[ur, dash, "X(x'{,}x)"] \arrow[rrrr, dash, "\rho(x'{,}y')"] & & & & y' \end{tikzcd}$$

- Composition with $Y \overset{\sigma}{\longrightarrow\!\!\!\circ\!\!\to} Z$: $(\sigma\cdot\rho)(x,z) = \inf_{y\in Y}(\rho(x,y)+\sigma(y,z))$
- Examples: $\mathrm{Met}_1 \longrightarrow \mathcal{R}_+\text{-Dist}$, $(\,X \xrightarrow{f} Y\,) \mapsto (\,X \overset{f_*}{\longrightarrow\!\!\!\circ\!\!\to} Y\,)$, $f_*(x,y) = Y(fx,y)$
- Making $\mathcal{R}_+$-Dist normed:

$$|\rho| := \sup_{x\in X}\inf_{y\in Y}\rho(x,y) \qquad \|\rho\| := \inf_{\varphi: X\to Y}\sup_{x\in X}\rho(x,\varphi x) \qquad \mathrm{AC} \Longrightarrow |\rho| = \|\rho\|$$

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Properties of $\mathsf{Cat}/\!/\mathcal{V}$, $\mathcal{V}$-normed categories *vs.* ordinary categories

$\mathsf{Cat}/\!/\mathcal{V}$ is

- topological over Cat
- locally presentable
- symmetric monoidal closed

$$\mathbb{X}\otimes\mathbb{Y}=\mathbb{X}\times\mathbb{Y},\quad |(f,f')|=|f|\otimes|f'| \qquad \mathbb{E}=\{*\to *\},\quad |*\to *|=\mathrm{k}$$

$$[\mathbb{X},\mathbb{Y}]=(\mathsf{Cat}/\!/\mathcal{V})(\mathbb{X},\mathbb{Y}),\quad |\alpha:F\to G|=\bigwedge_{x\in\mathrm{ob}\mathbb{X}}|\alpha_x|$$

$\mathcal{V}\longrightarrow 2$ induces $\mathsf{Cat}/\!/\mathcal{V}\longrightarrow\mathsf{Cat}/\!/2$

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with $\mathbb{X}_\circ:=\{f:\mathrm{k}\le|f|\}$ (à la Kelly)

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Set$//\mathcal{V}$ as a $\mathcal{V}$-normed category??

Every monoidal-closed category $\mathcal{W}$ becomes $\mathcal{W}$-enriched, *qua* its internal hom.

Q: What happens to $\mathcal{W} = \mathrm{Set}//\mathcal{V}$?

A: Obtain a category with $\mathcal{V}$-normed sets as objects and arbitrary maps as morphisms:

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Normed convergence of sequences

$s : \mathbb{N} \longrightarrow \mathbb{X}$ $\qquad x_o \xrightarrow{s_{0,1}} x_1 \xrightarrow{s_{1,2}} x_2 \longrightarrow \dots \longrightarrow x_m \xrightarrow{s_{m,n}} x_n \longrightarrow \dots$

$s_{n,n} = 1_{x_{n,n}}, \quad s_{m,n} \cdot s_{k,m} = s_{k,n}$

$s_{|N} = s|_{\uparrow N}$ $\qquad$ with $\uparrow N = \{n : n \geq N\}$

$x \cong \mathrm{ncolim}\, s$: (C1) $x \cong \mathrm{colim}\, s$ (in the ordinary category $\mathbb{X}$ with cocone $\gamma_n : x_n \to x$)

$$\begin{tikzcd} x_n \arrow[r, "\gamma_n"] \arrow[dr, "\alpha_n"'] & x \arrow[d, "f"] \\ & y \end{tikzcd} \qquad \alpha = (\alpha_n) \leftrightsquigarrow f \qquad \mathrm{Nat}(s, \Delta y) \cong \mathbb{X}(x, y)$$

(C2) $\forall y \in \mathbb{X} : (\mathrm{Nat}(s_{|N}, \Delta y) \to \mathbb{X}(x, y))_{N \in \mathbb{N}}$ is a colimit cocone in $\mathsf{Set}/\!/\mathcal{V}$

(C2) $\iff$ (C2a) $\qquad \mathrm{k} \leq \bigvee_N \bigwedge_{n \geq N} |\gamma_n|$

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Normed convergence of sequences

$s : \mathbb{N} \longrightarrow \mathbb{X}$ $\quad x_o \xrightarrow{s_{0,1}} x_1 \xrightarrow{s_{1,2}} x_2 \longrightarrow \dots \longrightarrow x_m \xrightarrow{s_{m,n}} x_n \longrightarrow \dots$

$s_{n,n} = 1_{x_{n,n}}, \quad s_{m,n} \cdot s_{k,m} = s_{k,n}$

$s_{|N} = s|_{\uparrow N}$ $\quad$ with $\uparrow N = \{n : n \geq N\}$

$x \cong \mathrm{ncolim}\, s$: (C1) $x \cong \mathrm{colim}\, s$ (in the ordinary category $\mathbb{X}$ with cocone $\gamma_n : x_n \to x$)

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Comparison with [Kubiś 2017]

Fact:

Existence granted,
a normed colimit is unique up to a k-isomorphism, *i.e.,* up to an isomorphism in $\mathbb{X}_\circ$.

Comparison:

For a norm on a category, Kubiś requires (when we extrapolate from $\mathcal{R}_+$ to a quantale $\mathcal{V}$) the additional condition

$$(\mathrm{S}^{\mathrm{op}}) \qquad |g \cdot f| \otimes |g| \leq |f|,$$

but for $x \cong \mathrm{ncolim}\, s$, he requires only (C1) and (C2a), not (C2b).

Unfortunately, as he observes himself, this does not make normed colimits unique up to a k–isomorphism.

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A word about symmetry

$$(\mathrm{S}) \qquad |f \cdot h| \otimes |h| \leq |f|$$

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For $\mathbb{X} = \mathrm{i}X, \ X \in \mathcal{V}$-Cat:

$$(\mathrm{S}) \iff X(x,y) = X(y,x) \iff (\mathrm{S}^{\mathrm{op}})$$

For $(\mathbb{X}, \mathcal{S}) \in \mathsf{Cat}//2$:

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Cauchy cocompleteness

$s : \mathbb{N} \to \mathbb{X}$ Cauchy $\quad :\iff \mathrm{k} \leq \bigvee_N \bigwedge_{n \geq N} |s_{m,n}|$

$\mathbb{X}$ Cauchy cocomplete $:\iff$ every Cauchy sequence in $\mathbb{X}$ has a normed colimit in $\mathbb{X}$

For $\mathbb{X} = \mathrm{i}X,\ X \in \mathcal{R}_+$-Cat $=$ Met$_1$:

$$s \text{ Cauchy} \iff \inf_N \sup_{n \geq m \geq N} X(x_m, x_n) = 0$$

$\iff$ s is forward Cauchy (see [Bonsangue, van Breugel, Rutten 1998])

$$x \cong \mathrm{ncolim}\, s \iff \forall y :\ X(x, y) = \inf_N \sup_{n \geq N} X(x_n, y)$$

$\iff$ x is a forward limit of s (in the sense of [BvBR 1998])

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A couple of cautionary notes

- A sequence with a normed colimit may not be Cauchy:

$$0 \xrightarrow{1} 1 \xrightarrow{1} 2 \xrightarrow{n-2} n \xrightarrow{\quad 0 \quad} \infty$$

- The constant sequence $\overline{e} = (\, x \xrightarrow{e} x \xrightarrow{e} x \longrightarrow \ldots)$ given by an idempotent e may not have a normed colimit, even if it has an ordinary colimit and is Cauchy:

$$\begin{aligned} \overline{e} \text{ has (ordinary) colimit in } \mathbb{X} &\iff e \text{ splits } (e = t \cdot r,\ r \cdot t = 1) \text{ in } \mathbb{X} \\ \overline{e} \text{ is Cauchy} &\iff \mathrm{k} \leq |e| \\ \overline{e} \text{ has normed colimit in } \mathbb{X} &\iff e \text{ splits such that } \mathrm{k} \leq |r|,\ \mathrm{k} \leq |t| \end{aligned}$$

Equivalent are for a $\mathcal{V}$-normed category $\mathbb{X}$:

- $\mathbb{X}_\circ$ is idempotent complete (*i.e.*, every idempotent in $\mathbb{X}_\circ$ splits in $\mathbb{X}_\circ$)
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The Boolean case: $\mathcal{V} = 2$

For a sequence s in $(\mathbb{X}, \mathcal{S}) \in \mathsf{Cat}//2$:

$$s \text{ is Cauchy} \iff \text{eventually all connecting maps are in } \mathcal{S}$$

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for all $f : x \to y$

$$f \in S \iff \text{eventually all } f \cdot \gamma_n \text{ are in } \mathcal{S}$$

$(\mathbb{X}, \mathcal{S})$ with $\mathbb{X} = \mathsf{Top}$ is Cauchy cocomplete for $\mathcal{S} = \{\text{monos}\}$ and $\mathcal{S} = \{\text{closed embed's}\}$,

but not for $\mathcal{S} = \{\text{embeddings}\}$ [Adámek, Hušek, Rosický, T 2023]:

$$\begin{tikzcd} (\downarrow n \times \mathbb{N})_{\mathrm{cof}} \arrow[r, "\gamma_n"] \arrow[dr, "f \cdot \gamma_n"'] & \operatorname{colim} s \arrow[d, "f = \mathrm{id} \notin \mathcal{S}"] \\ & (\mathbb{N} \times \mathbb{N})_{\mathrm{cof}} \end{tikzcd}$$

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Extending the real arithmetic to $[0, \infty]$

Put $e^{\infty} = \infty$ and $0 \cdot \infty = \infty$ (*i.e.* (-) $\cdot\, \infty$ preserves infima) and consider the adjunction

$$\begin{tikzcd} \mathcal{R}_+ = ([0,\infty], \geq, +, 0) \arrow[r, "\exp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{R}_\times = ([0,\infty], \geq, \cdot, 1) \arrow[l, bend left, "\log^\circ"] \end{tikzcd}$$

Internal hom $[\beta, \alpha]$:

$$\alpha \mathbin{\hat{-}} \beta = \max\{\alpha - \beta, 0\} \qquad \frac{\alpha}{\beta} = \inf\{\gamma : \alpha \leq \beta \cdot \gamma\}$$

Obtain

$$\frac{0}{0} = 0 \qquad \frac{\alpha}{0} = \infty\ (\alpha > 0) \qquad \frac{\alpha}{\infty} = \frac{\infty}{\infty} = 0$$

$$\log^\circ 0 = 0 \qquad \log^\circ \alpha = \max\{0, \log \alpha\}\ (0 < \alpha < \infty) \qquad \log^\circ \infty = \infty$$

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The normed categories SNVec_∞ and NVec_∞

A seminorm $\|\text{-}\| : X \to [0, \infty]$ on a vector space X must satisfy:

- $\|0\| = 0$
- $\|ax\| = |a|\,\|x\|$ $(a \in \mathbb{R}, a \neq 0)$
- $\|x + y\| \leq \|x\| + \|y\|$

With all linear maps as morphisms normed by

$$|X \xrightarrow{f} Y| := \sup_{x \in X} \log^{\circ}\left(\frac{\|fx\|}{\|x\|}\right)$$

one obtains the normed category SNVec_∞.
It contains the full subcategory NVec_∞ of separated seminormed spaces:

- $\|x\| = 0 \Longrightarrow x = 0$

Theorem

SNVec_∞ *is Cauchy cocomplete, but* NVec_∞ *is not.*

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Brief comment on the proof of the Theorem

The proof is harder than one may have expected, although the starting point seems clear:

For a Cauchy sequence $s = (X_m \xrightarrow{s_{m,n}} X_n)_{m \leq n}$, form the colimit $(X_n \xrightarrow{\gamma_n} X)_n$ in Vec and put

$$\|x\| := \sup_{N \in \mathbb{N}} \inf_{n \geq N} \inf_{z \in \gamma_n^{-1} x} \|z\|_n \quad (x \in X)$$

Now prove that that this makes X a seminormed space and verify conditions (C2a), (C2b).

The negative assertion about NVec_∞ follows more easily, with the help of

$$\mathbb{R} = \mathbb{R}_1 \longrightarrow \mathbb{R}_{\frac{1}{2}} \longrightarrow \mathbb{R}_{\frac{1}{3}} \longrightarrow \ldots \longrightarrow \mathrm{colim} = 0$$

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A final remark on NVec_∞

Call a linear map $f : X \to Y$ of seminormed vector spaces a 0-to-0 morphism if

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holds. This defines the wide subcategory SNVec_{00} of SNVec_∞.

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The normed category NVec_∞ is a full reflective subcategory of SNVec_{00} (not of SNVec_∞). It has colimits of all those Cauchy sequences whose normed colimit in SNVec_∞ is also a colimit in the ordinary category SNVec_{00}.

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Banach spaces?

An existing normed colimit in NVec_∞ of a Cauchy sequence of isometric embeddings of Banach spaces may not be Banach:

$$\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow \dots \longrightarrow \bigoplus_n \mathbb{R}^n$$

Indeed, the Cauchy sequence $(x_n)_n$,

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No linear structure: Is Met_∞ Cauchy cocomplete?

Met_∞:

- Objects: (Lawvere) metric spaces
- Morphisms: all maps, normed by

$$|X \xrightarrow{\varphi} Y| := \sup_{x,x' \in X} \log^\circ\left(\frac{Y(\varphi x, \varphi x')}{X(x,x')}\right)$$

$\mathcal{V}$-Lip:

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$$|X \xrightarrow{\varphi} Y| := \bigwedge_{x,x' \in X} [X(x,x'), Y(\varphi x, \varphi x')]$$

Get Met_∞ from $\mathcal{R}_\times$-Lip via change of base:

$$\begin{tikzcd} \mathcal{R}_+ \arrow[r, "\exp", shift left] \arrow[r, phantom, "\perp"] & \mathcal{R}_\times \arrow[l, bend left, "\log^\circ"] \end{tikzcd}$$

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Yes: Met_∞ is Cauchy cocomplete!

Recall “totally below”:

$$u \ll \bigvee_{i \in I} v_i \iff \exists\, i \in I : u \ll v_I$$

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$\mathcal{V}$-Lip *is Cauchy cocomplete if*

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- k *is approximated from totally below:* $\bigvee \Downarrow \mathrm{k} = \mathrm{k}$;
- $\otimes$ *preserves* $\ll$: $\quad (u \ll v,\ w > \perp \Longrightarrow u \otimes w \ll v \otimes w)$.

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Theorem

For every small $\mathcal{V}$-normed category $\mathbb{X}$, the $\mathcal{V}$-normed presheaf category

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is Cauchy cocomplete, provided that $\mathcal{V}$ satisfies

- (A) k *is approximated from totally below:* $\bigvee \Downarrow\mathrm{k} = \mathrm{k}$;

  OR

- (B) k $\wedge$*-distributes over arbitrary joins:* $\mathrm{k} \wedge \bigvee_i v_i = \bigvee_i \mathrm{k} \wedge v_i$.

Corollary

The presheaf category $[\mathbb{X}, \mathsf{Set}||\mathcal{V}]$ is Cauchy cocomplete whenever $\mathcal{V}$ is integral ($\mathrm{k} = \top$), or $\mathcal{V}$ is a frame, or $\mathcal{V}$ is constructively completely distributive.

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Further remarks on the Theorem

- The proof is much harder than expected!
- We don't know of a quantale $\mathcal{V}$ for which Set$||\mathcal{V}$ fails to be Cauchy cocomplete!
- Conditions (A) and (B) are independent of each other:

(B) $\not\Rightarrow$ (A):

For any infinite set X and its cofinite topology $\mathcal{O}(X)$, consider $(\mathcal{O}(X), \subseteq, \cap, X)$.

Then any open set U with $U \ll X$ must be empty since, otherwise, we have

$$X = \bigcup_{x \in U} X \setminus \{x\},$$

whereas no $x \in U$ allows $U \subseteq X \setminus \{x\}$. Consequently, (A) is violated in $\mathcal{O}(X)$.

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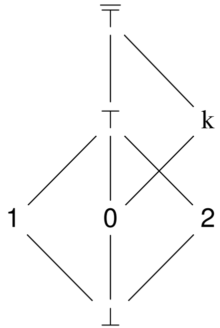
(A) $\not\Rightarrow$ (B) [Gutiérrez-García, Höhle 2024]



$\otimes$	$\overline{\top}$	k	$\top$	0	1	2	$\perp$
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2	$\top$	2	$\top$	2	0	1	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$

$$\mathrm{k} \wedge (1 \vee 2) = 0 \neq \perp = (\mathrm{k} \wedge 1) \vee (\mathrm{k} \wedge 2)$$

Reminders: weighted colimits, distributors, accessible presheaves

$F : \mathbb{A} \to \mathbb{X}$, $\varphi : \mathbb{A}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$ $\mathcal{V}$-normed functors of $\mathcal{V}$-normed categories $\mathbb{A}, \mathbb{X}$ ($\mathbb{A}$ small),

also written as composable $\mathcal{V}$-distributors: $F^* : \mathbb{X} \longrightarrow\!\!\!\!\circ\!\!\!\to \mathbb{A}$, $\varphi : \mathbb{A} \longrightarrow\!\!\!\!\circ\!\!\!\to \mathbb{E}$

$$\begin{aligned} x \cong \mathrm{colim}^{\varphi} F \iff & \mathbb{X}(x,y) \cong \mathrm{Nat}(\varphi, \mathbb{X}(F-,y)) \text{ naturally in } y \\ \iff & x \cong \mathrm{colim}^{\varphi \cdot F^*} \mathrm{id}_{\mathbb{X}} \\ \iff & : \text{“ } x \text{ is a weighted colimit of } \varphi \cdot F^* \text{ ”} \end{aligned}$$

After [Kelly-Schmitt 2005]:

$\psi : \mathbb{X}^{\mathrm{op}} \to \mathsf{Set}||\mathcal{V}$ accessible $: \iff \psi = \varphi \cdot F^*$ for some F, φ as above

$\mathcal{P}\mathbb{X} :=$ full normed subcategory of $[\mathbb{X}^{\mathrm{op}}, \mathsf{Set}||\mathcal{V}]$ of all accessible presheaves on $\mathbb{X}$

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Normed colimits as weighted colimits

Proposition

If $\mathcal{V}$ satisfies condition (A) or (B), then for every $\mathcal{V}$-normed category $\mathbb{X}$, $\mathcal{P}\mathbb{X}$ is Cauchy cocomplete.

For a Cauchy sequence s in the $\mathcal{V}$-normed category $\mathbb{X}$, form

$$\varphi_s \cong \mathrm{ncolim}\,(\mathbb{N} \xrightarrow{s} \mathbb{X} \xrightarrow{\mathbf{y}_{\mathbb{X}}} \mathcal{P}\mathbb{X})$$

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$$x \cong \mathrm{ncolim}\, s \iff x \cong \mathrm{colim}^{\varphi_s}\mathrm{id}_{\mathbb{X}}$$

Corollary

$\mathbb{X}$ Cauchy cocomplete $\iff$ $\mathbb{X}$ has weighted colimits for all $F : \mathbb{A} \to \mathbb{X}$, $\varphi : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}||\mathcal{V}$, with $\mathbb{A}$ countable and φ a normed colimit of a Cauchy sequence of representables in $\mathcal{P}\mathbb{A}$.

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Cauchy cocompletion (à la [Kelly, Schmitt 2005])

$\mathcal{V}$ continues to satisfy (A) or (B).

Let Φ be the class of weights used in the Corollary, so that

$$\mathbb{X} \text{ is Cauchy cocomplete} \iff \mathbb{X} \text{ is } \Phi\text{-cocomplete} .$$

Let $\Phi(\mathbb{X})$ be the least full replete $\mathcal{V}$-normed subcategory of $\mathcal{P}\mathbb{X}$ closed under Φ-colimits.

Theorem

For every $\mathcal{V}$-normed category $\mathbb{X}$ and every Cauchy cocomplete $\mathcal{V}$-normed category $\mathbb{Y}$, the composition with the restricted Yoneda embedding $\mathbf{y}_{\mathbb{X}} : \mathbb{X} \to \Phi(\mathbb{X})$ defines an equivalence

$$(\Phi\text{-COCTS})(\Phi(\mathbb{X}), \mathbb{Y}) \to (\mathrm{CAT}//\mathcal{V})(\mathbb{X}, \mathbb{Y}) .$$

That is, $\Phi(-)$ provides a left biadjoint to the inclusion 2-functor Φ-COCTS $\to \mathrm{CAT}//\mathcal{V}$.

The equivalence restricts to $(\Phi\text{-Cocts})(\Phi(\mathbb{X}), \mathbb{Y}) \to (\mathrm{Cat}//\mathcal{V})(\mathbb{X}, \mathbb{Y})$ for small $\mathbb{X}$ and $\mathbb{Y}$.

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Banach's Fixed Point Theorem

Let $\mathbb{X}$ be ($\mathcal{R}_+$-)normed and $F : \mathbb{X} \to \mathbb{X}$ contractive: there is $\ell < 1$ with $|Fh| \leq \ell |h|$ for all h.

Suppose we have some $f : x \to Fx$ with $|f| < \infty$. Just like for metric spaces, the sequence

$$s_f = (\ x \xrightarrow{f} Fx \xrightarrow{Ff} F^2x \xrightarrow{F^2f} F^3x \xrightarrow{F^3f} \ldots\ )$$

is Cauchy. Would its colimit be a "fixed point" of F?

Theorem

Let $\mathbb{X}$ be Cauchy cocomplete with some f as above. If the contractive endofunctor

- *F preserves $y \cong \mathrm{colim}\, s_f$, then the canonical $\bar{f} : y \to Fy$ is an isom. with $|\bar{f}| = 0$;*
- *F preserves $y \cong \mathrm{ncolim}\, s_f$, then the canonical $\bar{f} : y \to Fy$ is a 0-isom.: $|\bar{f}| = 0 = |\bar{f}^{-1}|$.*

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- What about complete metric spaces? Banach spaces? Hilbert spaces?
- Find a quantale $\mathcal{V}$ such that $\mathsf{Set}||\mathcal{V}$ fails to be Cauchy cocomplete!
- Is $\mathcal{V}$-Dist with the Hausdorff norm Cauchy cocomplete?
- Describe the Cauchy cocompletion of a a normed category “more constructively”?
- Why not directed or filtered systems instead of just sequences? Relevant examples?
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