Local mean dimension theory for sofic group actions

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Based on joint work with Felipe García-Ramos.

arXiv:2401.08440

38th Summer Conference on Topology and its Applications
(SUMTOPO 2024)

University of Coimbra

8-12.07.2024

Standing notation I

Objects:

- Topological dynamical systems (t.d.s) (G, X), where:
  X - compact metrizable space.

  G - topological group.

  $G \times X \to X$ continuous s.t $ex = x$ and $h(gx) = (hg)x$.
- When $G = \mathbb{Z}$, we write (X, T) where T is the generator of the $\mathbb{Z}$-action.
- $Y \subset X$ closed and G-invariant ($GY = Y$) is called a subsystem.
- (G, X) is called minimal if only subsystems are $Y = \emptyset$ and $Y = X$.
- (G, X) is called aperiodic (free) if $\exists x \in X\; gx = x$ implies $g = e$.

Standing notation II

Morphisms:

$$\phi : (G, X) \to (G, Y)$$

where, $\phi : X \to Y$ - Equivariant continuous mapping ($\phi(gx) = g\phi(x)$, $\forall x \in X, g \in G$):

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

Special morphisms:

ϕ is onto $\overset{\triangle}{\Leftrightarrow}$ $\phi : (G, X) \twoheadrightarrow (G, Y)$ $\overset{\triangle}{\Leftrightarrow}$ ϕ is an extension or factor map

ϕ is injective $\overset{\triangle}{\Leftrightarrow}$ $\phi : (G, X) \hookrightarrow (G, Y)$ $\overset{\triangle}{\Leftrightarrow}$ ϕ is an (dynamical) embedding

Embedding dimension

- For any compact metrizable space X there is a (topological) embedding into the Hilbert cube: $\phi : X \hookrightarrow [0,1]^{\mathbb{N}}$.
- Let (X, T) be a t.d.s. There is a (dynamical) embedding by the orbit-map:

$$\begin{aligned} \Phi : (X, T) &\hookrightarrow (([0,1]^{\mathbb{N}})^{\mathbb{Z}}, \mathrm{shift}) \\ x &\mapsto (\phi(T^k x))_{k\in\mathbb{Z}} \end{aligned}$$

- Under which conditions is there an embedding into a d-cubical shift:

$$(X, T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, \mathrm{shift})\,(d \in \mathbb{N})?$$

- Define the embedding dimension:

$$\mathrm{edim}(X, T) = \min\{d \in \mathbb{N} \cup \{\infty\} \,|\, (X, T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, \mathrm{shift})\}$$

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Question (Auslander, 1988)

Let (X, T) be minimal. Is $\mathrm{edim}(X, T) = 1$*?*

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Mean dimension

- Let $f : X \to Y$ be a continuous map and $\varepsilon > 0$. Let d be a compatible metric on X. The map f is called an ε-embedding if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for all $y \in Y$.
- Let $\operatorname{widim}_\epsilon(X, d)$ be the minimal integer $n \geq 0$ such that there exist an n-dimensional simplicial complex P and an ε-embedding $f : X \to P$.
- (Lebesgue) $\dim(X) = \lim_{\epsilon \to 0} \operatorname{widim}_\epsilon(X, d)$
- $d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y)$
- (Gromov) $\boxed{\operatorname{mdim}(X, T) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\operatorname{widim}_\epsilon(X, d_n)}{n}}$

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Topological definition of mean dimension

- $\alpha = \{U_1, U_2, \dots U_m\}$ is a finite open cover of X.
- $\mathrm{ord}(\alpha) = \max_{x \in X} \sum_{U \in \alpha} 1_U(x) - 1$
- A (finite) open cover β of X refines a (finite) open cover α of X if for every $V \in \beta$ there is a $U \in \alpha$ such that $V \subset U$.
- $\mathrm{D}(\alpha) = \min_{\beta \succ \alpha} \mathrm{ord}(\beta)$
- $\dim(X) = \sup_\alpha \mathrm{D}(\alpha)$
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Mean dimension obstruction for embedding

- $\mathrm{mdim}(([0,1]^d)^{\mathbb{Z}}, \mathrm{shift}) = d$
- If $Y \subset X$ is a subsystem then $\mathrm{mdim}(Y,T) \leq \mathrm{mdim}(X,T)$
- $\mathrm{edim}(X,T) \geq \mathrm{mdim}(X,T)$
- Lindenstrauss and Weiss constructed a minimal system (X,T) with $\mathrm{mdim}(X,T) > 1$, so $\mathrm{edim}(X,T) > 1$.

Theorem (G-Tsukamoto, 2020)

If (X,T) is (an extension of) an aperiodic minimal system and $\mathrm{mdim}(X,T) < \frac{d}{2}$, then $(X,T) \hookrightarrow (([0,1]^d)^{\mathbb{Z}}, \mathrm{shift})$.

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Local theory of mean dimension

Definition

- An open cover of X is standard if it is composed of two non-dense open sets.
- An open cover (U, V) distinguishes $(x, y) \in X \times X$ if $x \notin \overline{V}$ and $y \notin \overline{U}$. Such a cover is always standard.
- Conversely, every standard cover distinguishes some $(x, y) \in X \times X$.
- Let (X, T) be a t.d.s. A pair $(x, y) \in X \times X$ is said to be a mean dimension pair if for every standard open cover, α, which distinguishes (x, y), it holds

$$\operatorname{mdim}(\alpha) > 0.$$

Theorem (García-Ramos & G, 2024)

$\operatorname{mdim}(X, T) > 0$ *iff there are mean dimension pairs.*

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- Let (X, T) be a t.d.s. A pair $(x, y) \in X \times X$ is said to be a mean dimension pair if for every standard open cover, α, which distinguishes (x, y), it holds
$$\mathrm{mdim}(\alpha) > 0.$$

Theorem (García-Ramos & G, 2024)

$\mathrm{mdim}(X, T) > 0$ *iff there are mean dimension pairs.*

Completely positive mean dimension systems

Definition

(X, T) is a completely positive mean dimension (CPMD) system if every non-trivial factor of (X, T) has positive mean dimension.

Theorem (Lindenstrauss-Weiss 2000)

$(([0,1]^d)^{\mathbb{Z}}, \mathrm{shift})$ *is a completely positive mean dimension (CPMD) system.*

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Essentially the Brouwer fixed point theorem... $\square$

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Analytic and coanalytic sets of Polish spaces

Definition

A Polish space is a separable topological space that can be metrized using a complete metric. A subset of a Polish space is analytic if it is the continuous image of a Borel subset of a Polish space and coanalytic if it is the complement of an analytic set.

Basic Facts:

All Borel subsets of a Polish space are both analytic and coanalytic. Moreover, if a set is both analytic and coanalytic, then it is Borel. However, in every uncountable Polish space there are analytic, and hence coanalytic, sets which are not Borel.

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Parameterizing all dynamical systems

- Let (X, T) be a t.d.s, then $(X, T) \hookrightarrow (Q^{\mathbb{Z}}, \mathrm{shift})$, where $Q = [0, 1]^{\mathbb{N}}$ is the Hilbert cube.
- Let $\mathcal{S}(Q) = \{X \subset Q^{\mathbb{Z}} : X \text{ is closed, non-empty and shift-invariant}\}$ equipped with the Hausdorff metric. This is a Baire space which parameterizes all dynamical systems.
- $$\mathcal{S}_{+}(Q) = \{X \in \mathcal{S}(Q) : (X, \mathrm{shift}) \text{ is a CPMD system}\}.$$

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A coanalytic subset $\mathcal{A}$ of a Polish space $\mathcal{X}$ is complete coanalytic if for every coanalytic set $\mathcal{B}$ of a Polish space $\mathcal{Y}$, there exists a Borel function $f : \mathcal{Y} \to \mathcal{X}$ such that $f^{-1}(\mathcal{A}) = \mathcal{B}$.

Theorem (García-Ramos & G, 2024)

The set $\mathcal{S}_+(Q)$ is a complete coanalytic subset of $\mathcal{S}(Q)$, in particular not Borel.

Remark

The theorem is true in the generality of sofic group actions.

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There exists a continuous map $\psi\colon K(I) \to \mathcal{S}(I)$ such that $\psi(B)$ is a CPMD system if and only if B is countable.

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For $b \in B$, let $b^{\mathbb{Z}}$ be the point $x \in I^G$ such that $x_i = b$ for all $i \in \mathbb{Z}$.
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