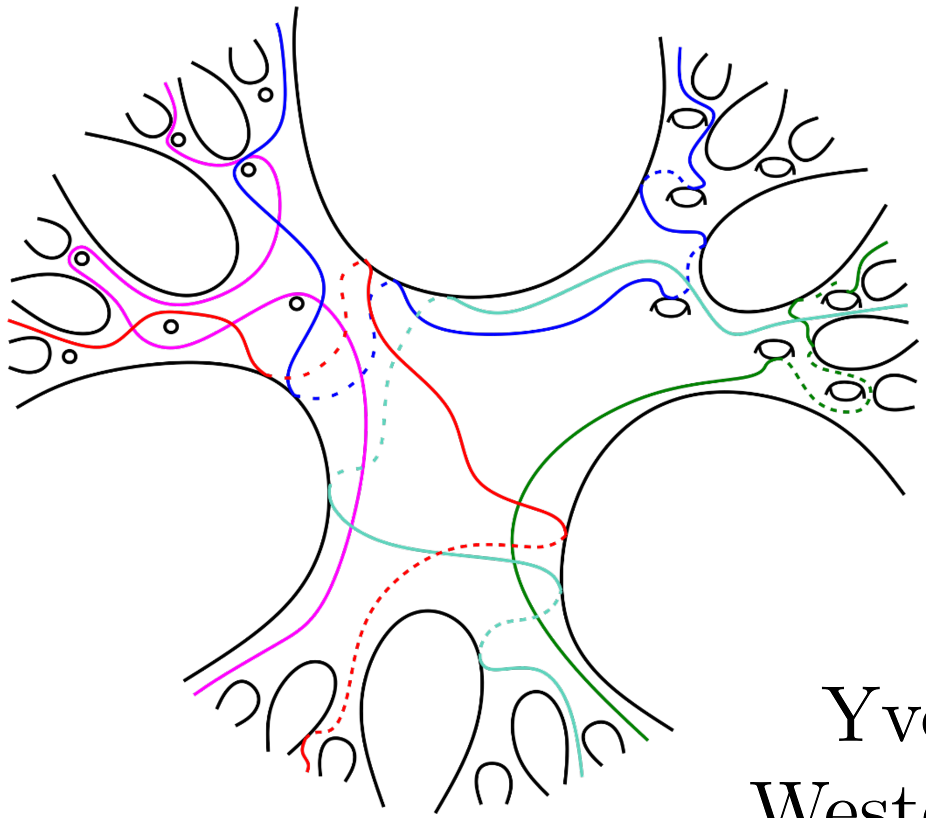
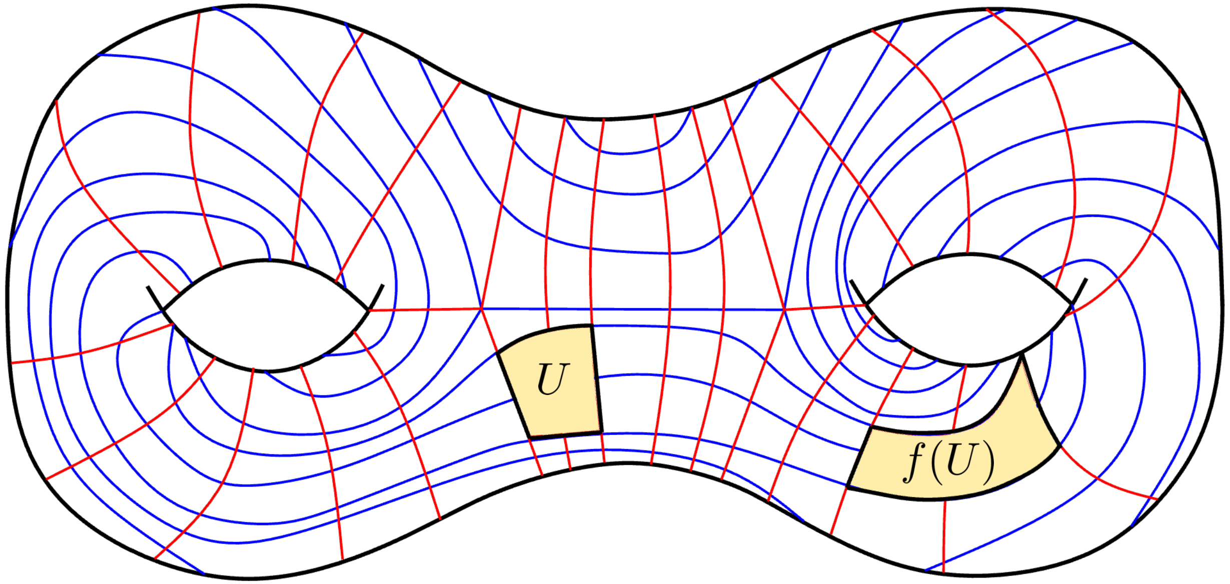
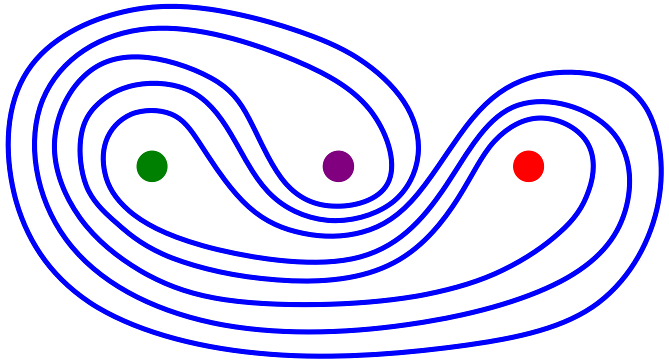
Pseudo-Anosov Homeomorphisms

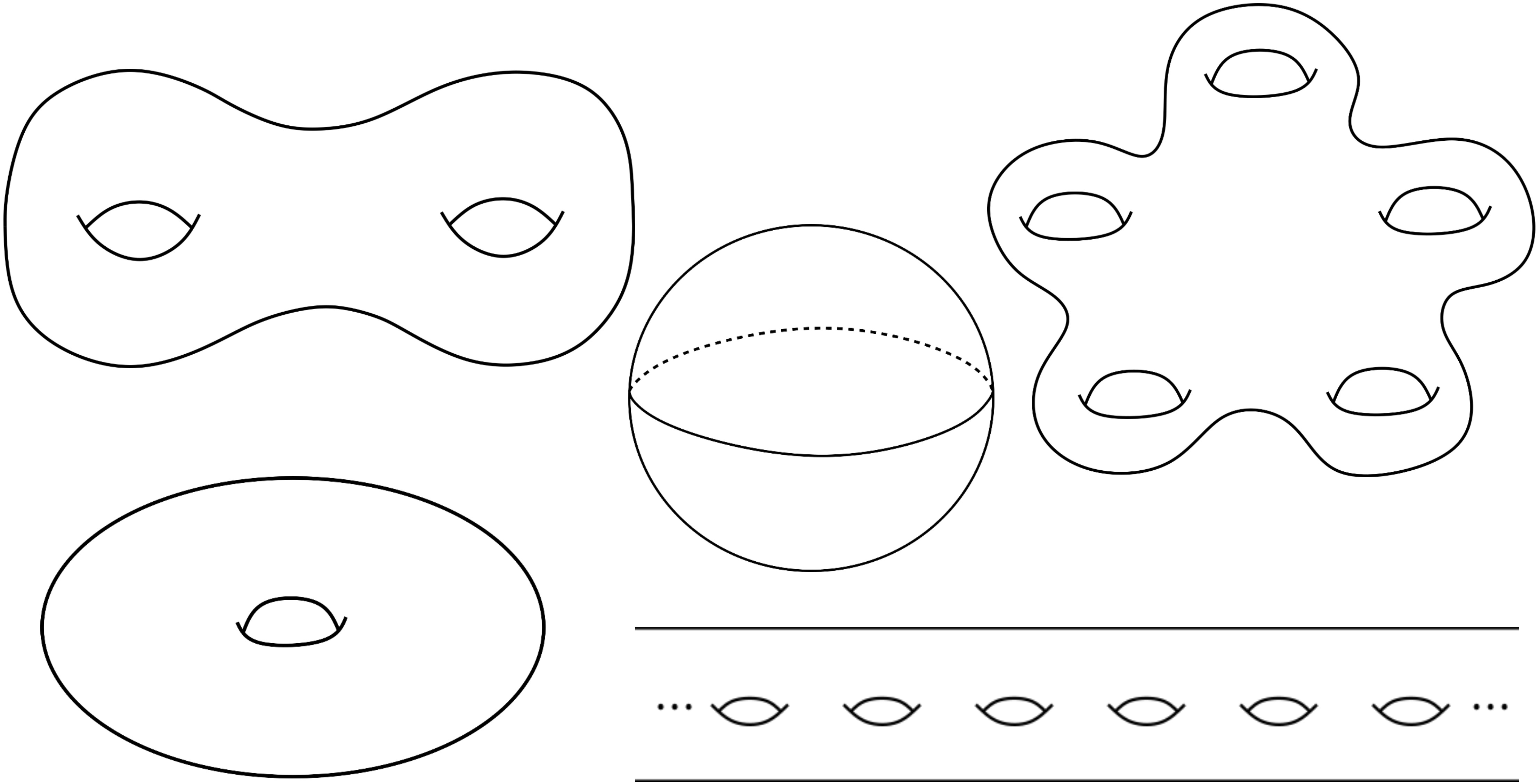




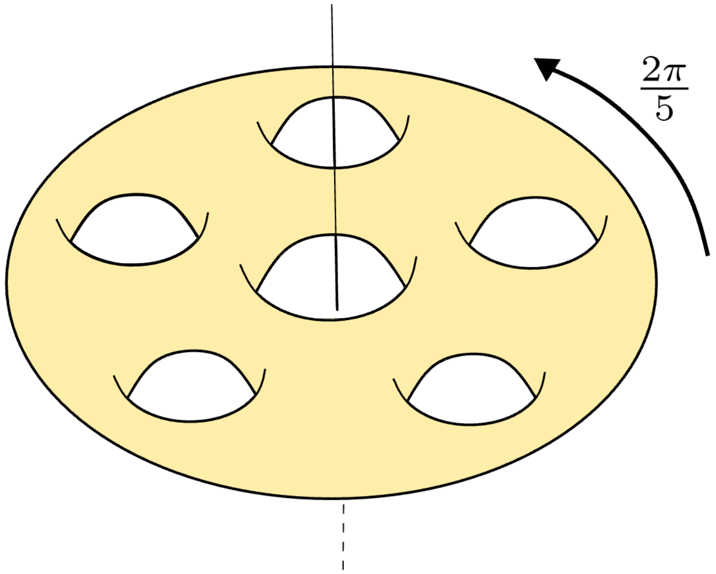
Yvon Verberne
Western University



Surfaces: Two-dimensional manifolds.



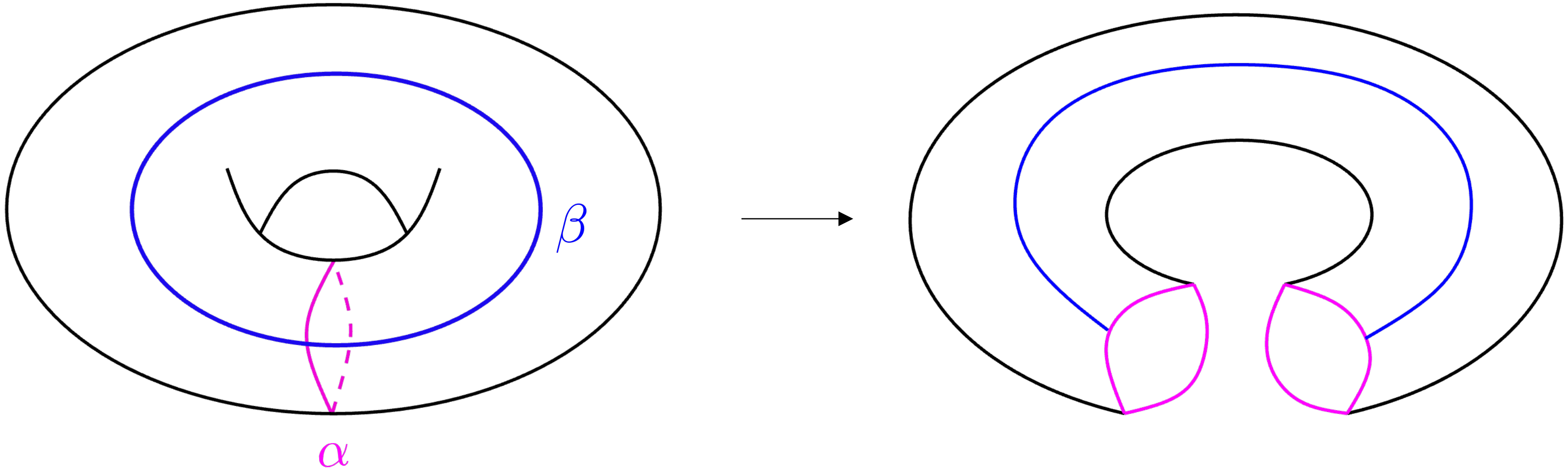
$$\mathrm{MCG}(S) = \mathrm{Homeo}(S)/\mathrm{Homotopy}$$



“Group of symmetries of a surface”

Generating Set

Dehn (1938) – Lickorish (1964): The mapping class group is generated by finitely many Dehn twists.



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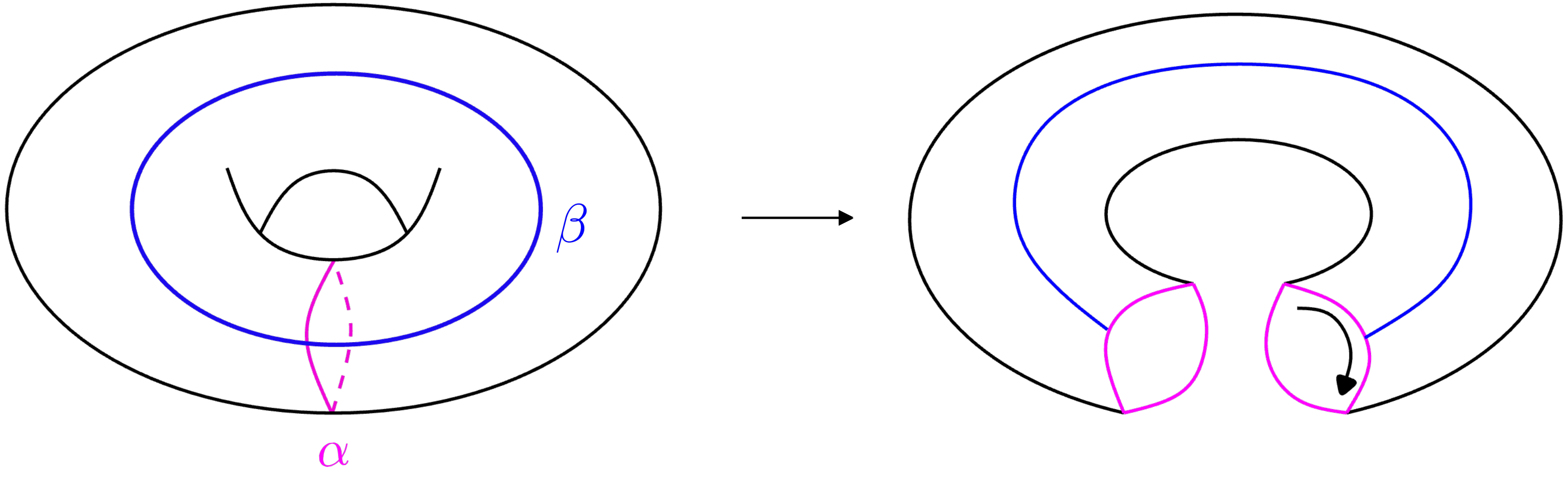

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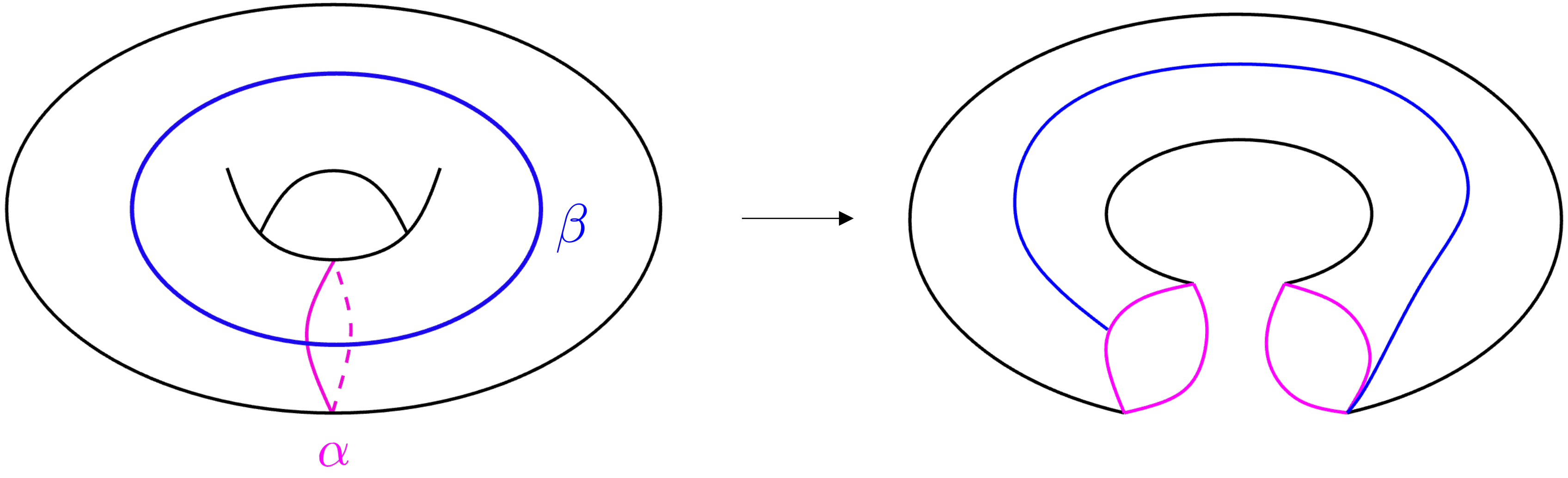

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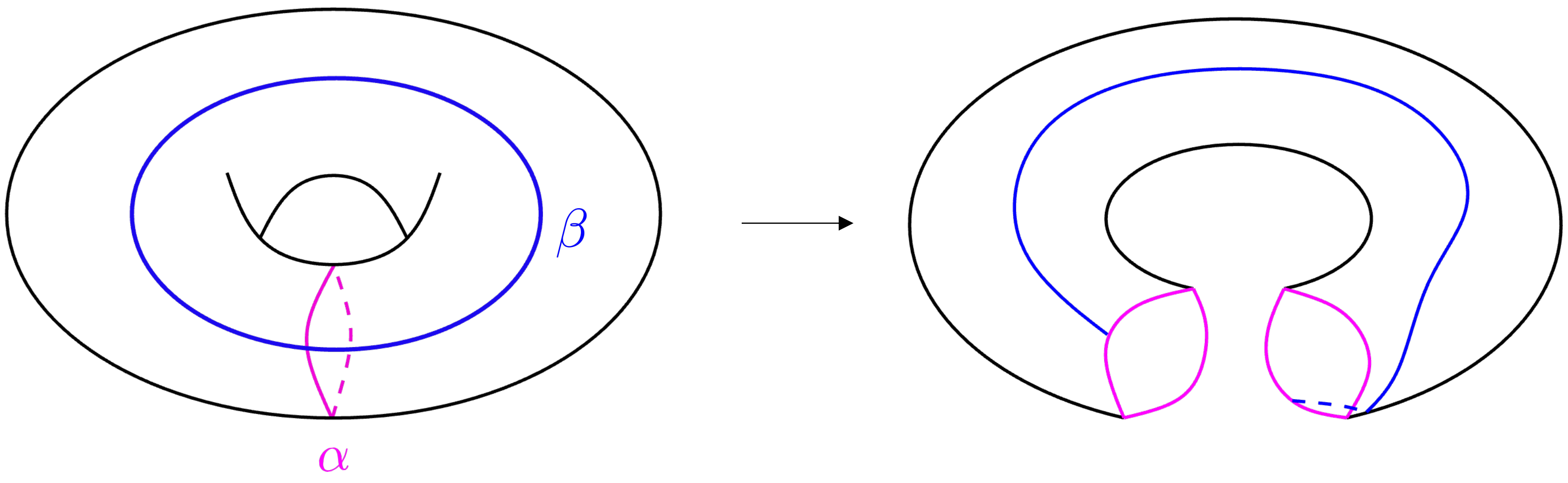

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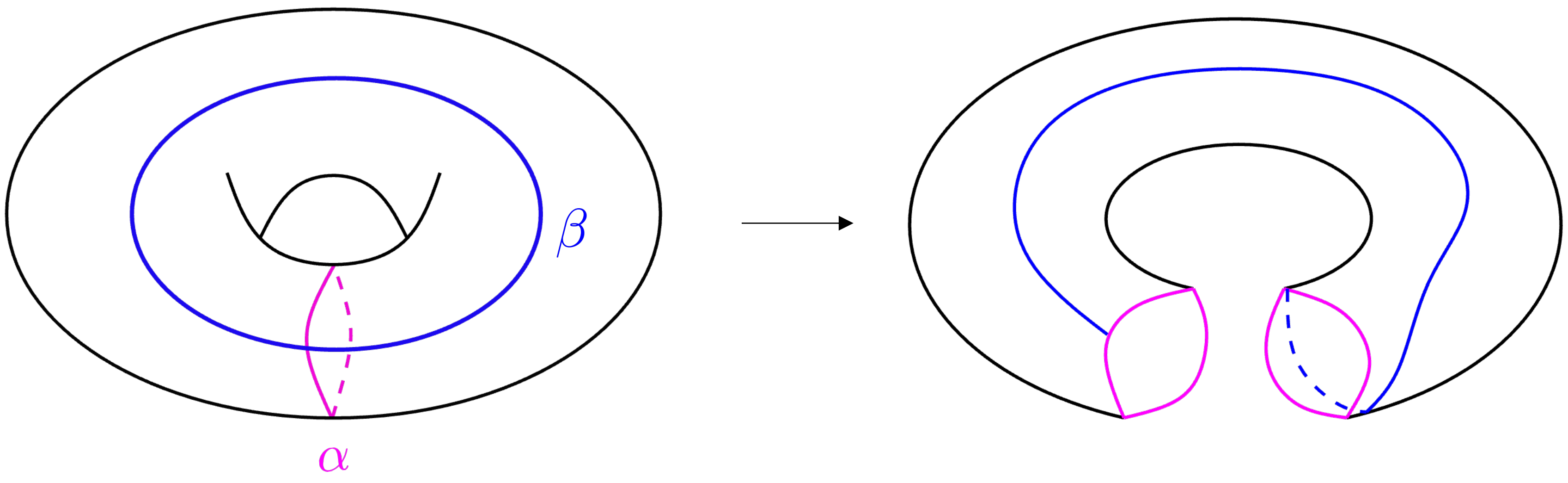

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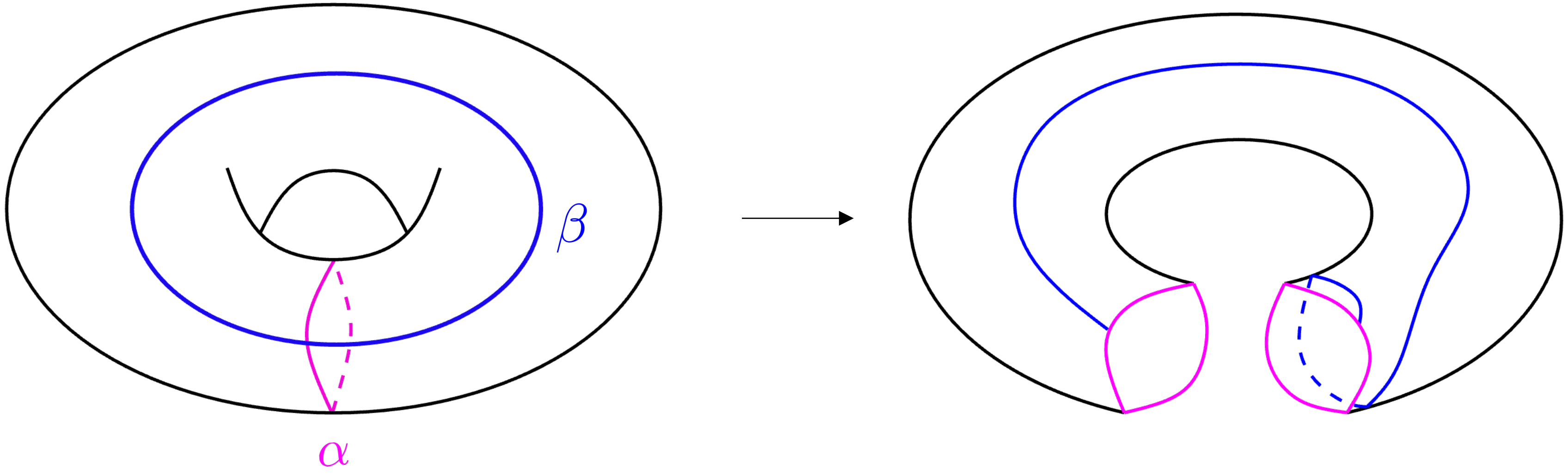

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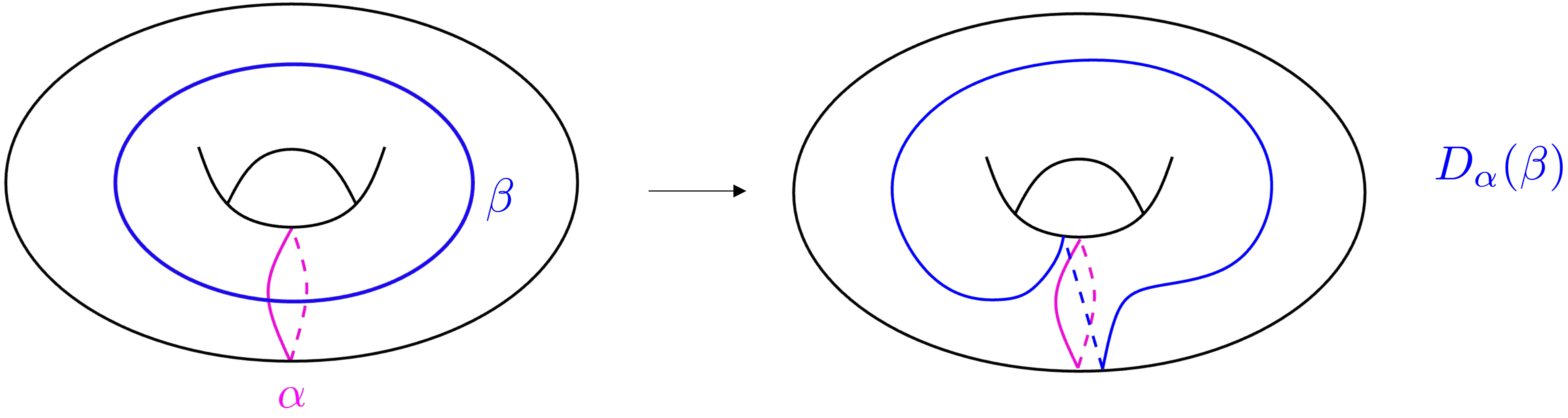
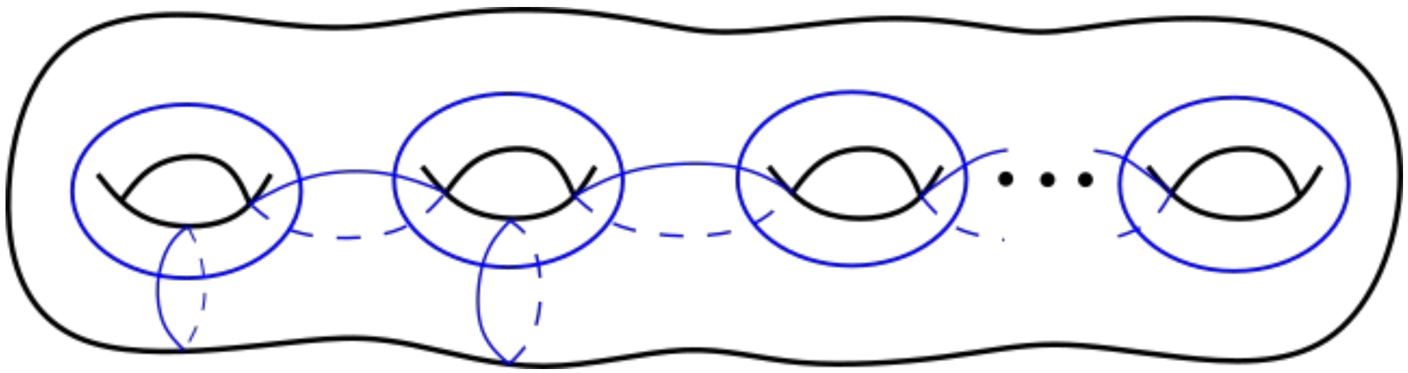


Generating Set

Dehn (1938) – Lickorish (1964): The mapping class group is generated by finitely many Dehn twists.

Humphries (1979): Require twists about $2g + 1$ curves for a surface of genus g.



$\mathrm{MCG}(S)^{ab}$ is Trivial for $g \geq 3$ (Harer [1983])

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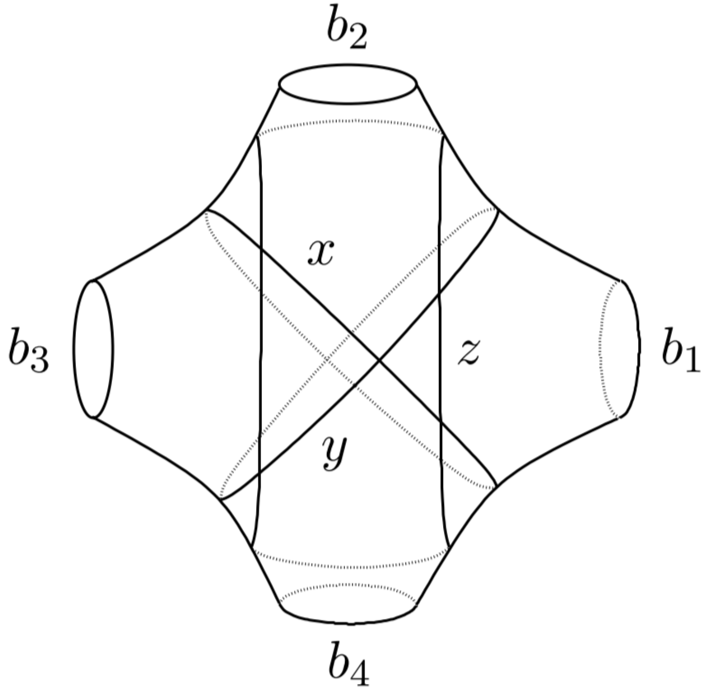

Lantern Relation:

$$D_x D_y D_z = D_{b_1} D_{b_2} D_{b_3} D_{b_4}$$

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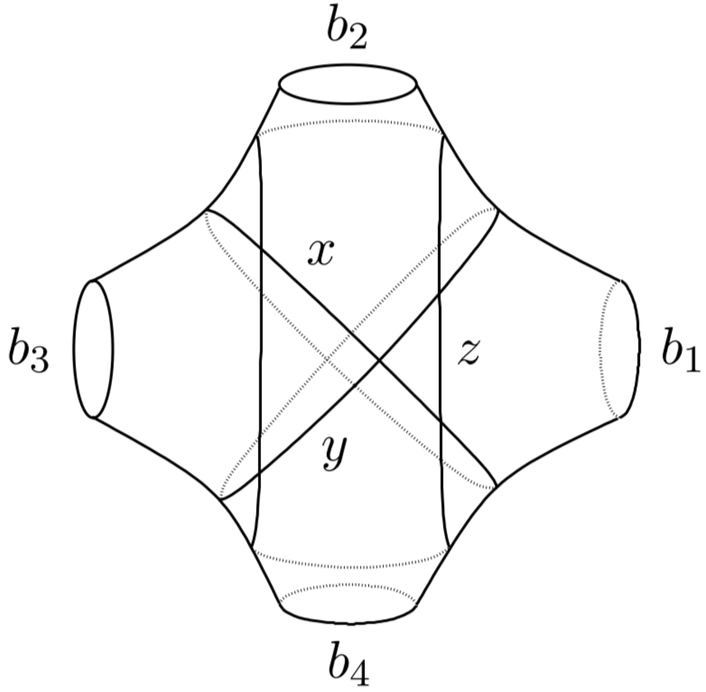

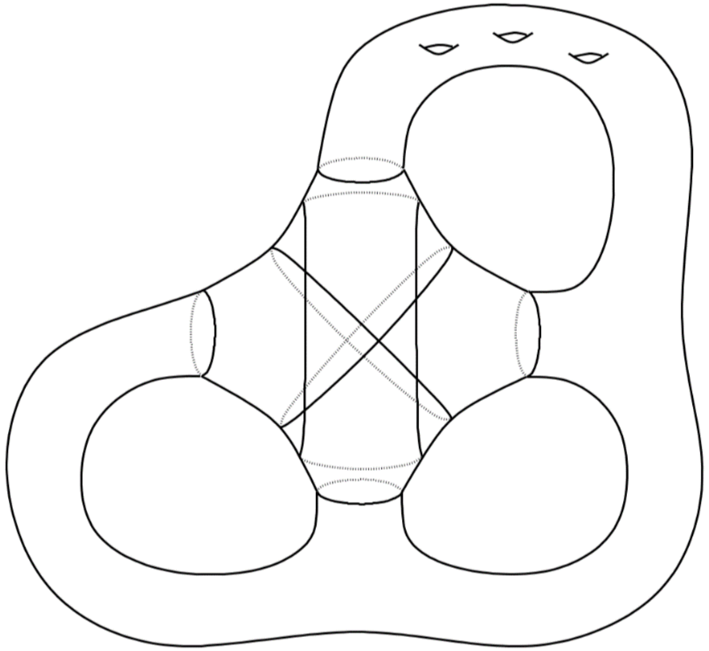

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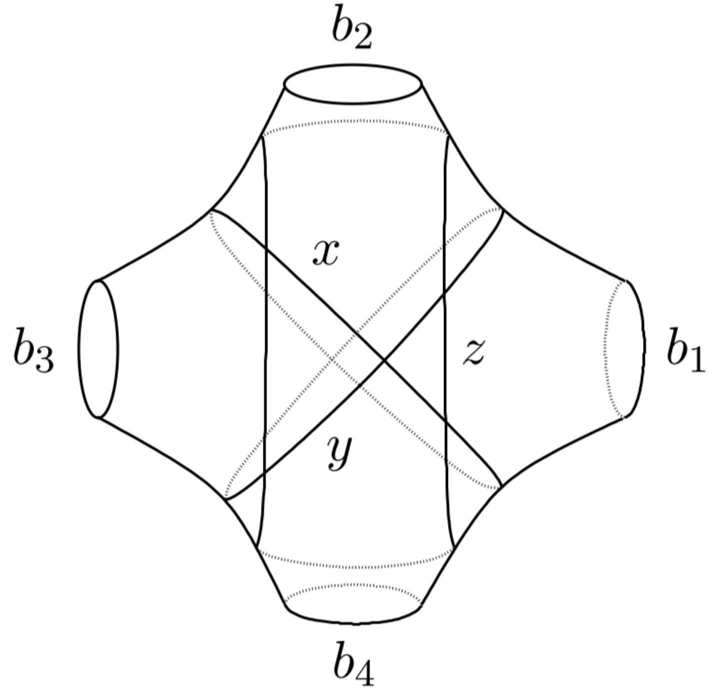

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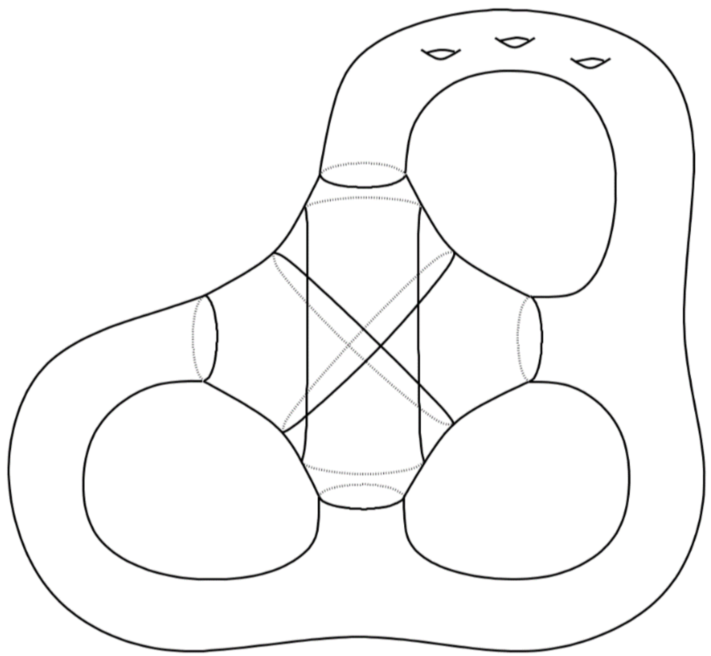
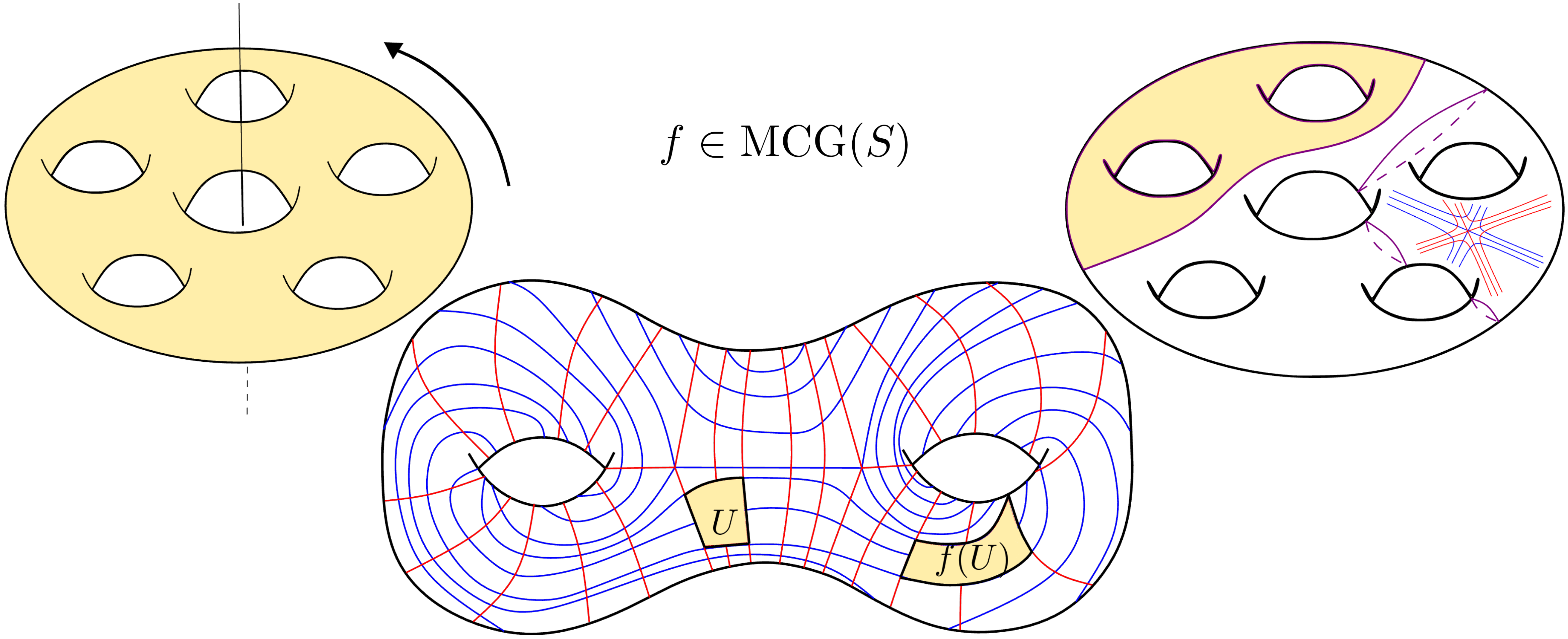


$\Longrightarrow h^3 = h^4$
i.e. h is trivial

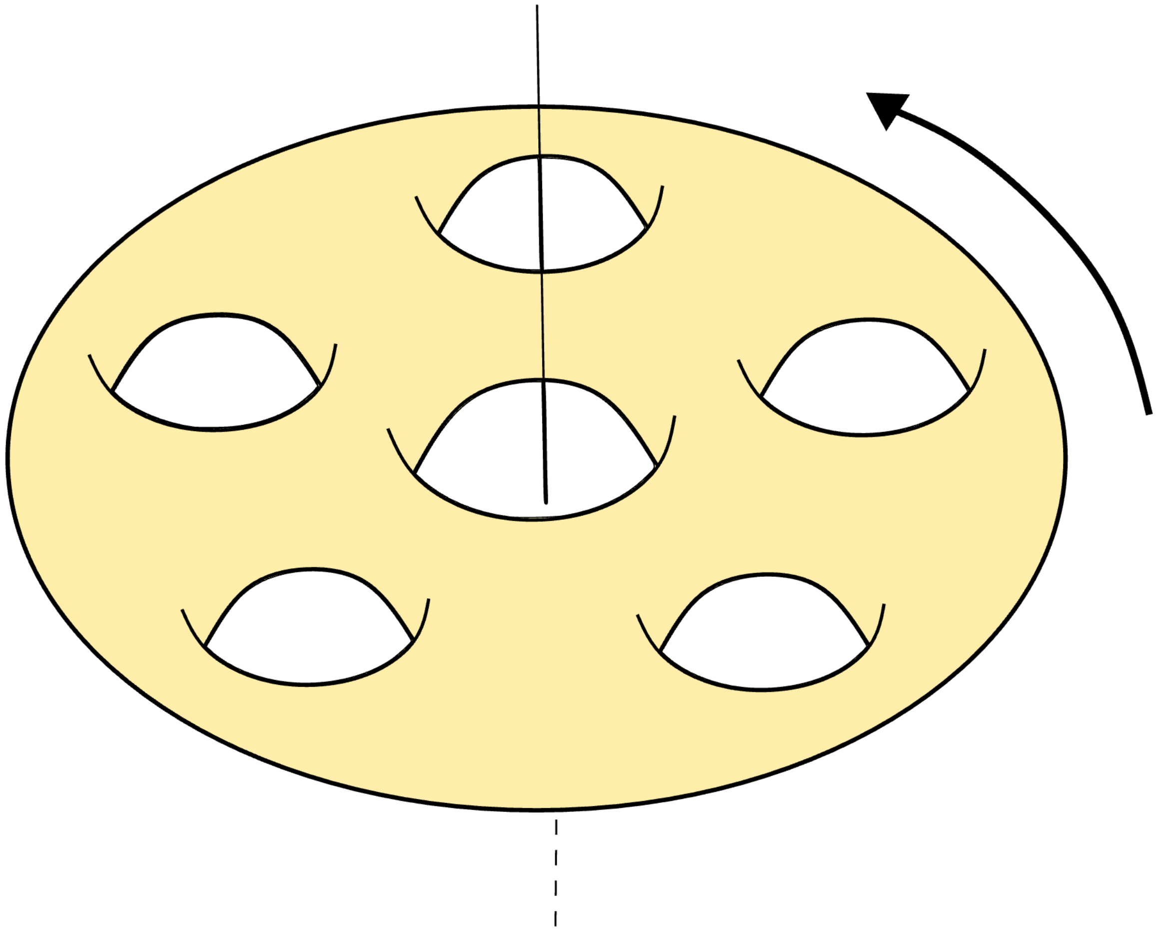
Types of Mapping Classes

Nielsen–Thurston Classification

$f \in \mathrm{MCG}(S)$

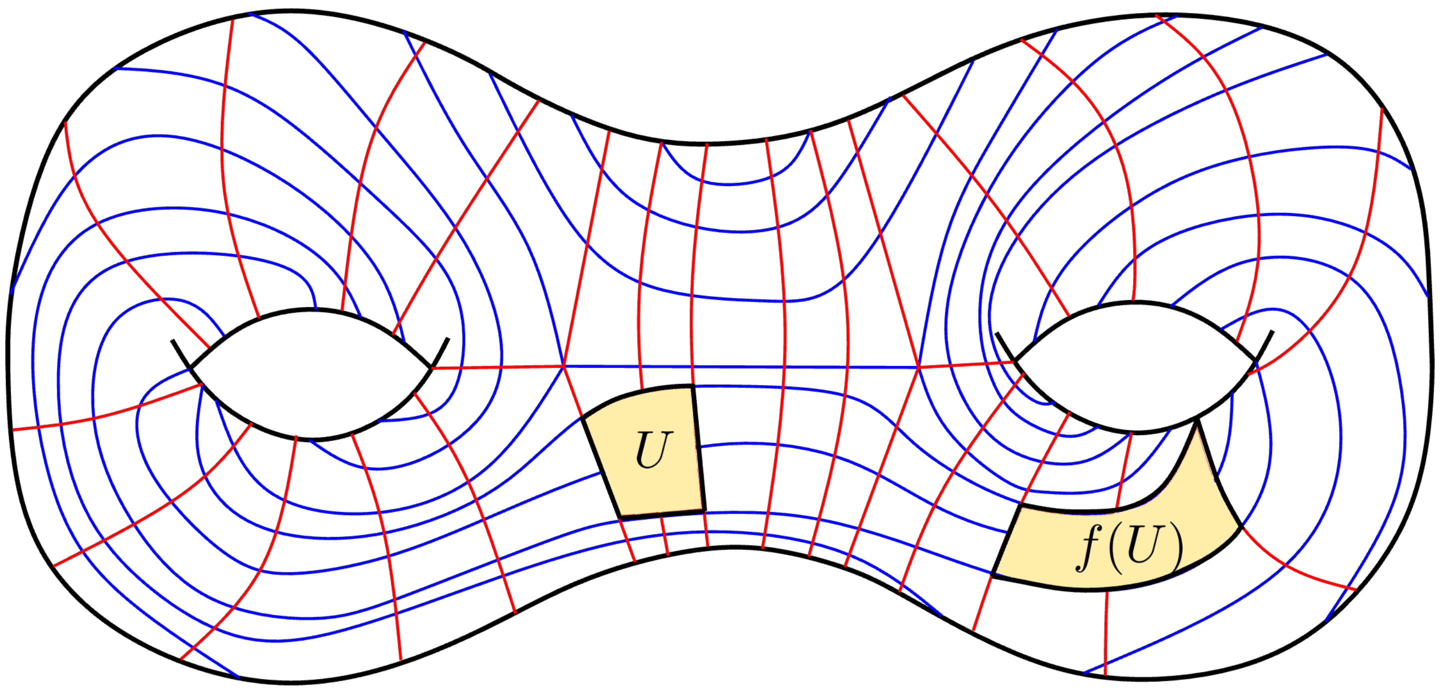


Periodic



f has finite order

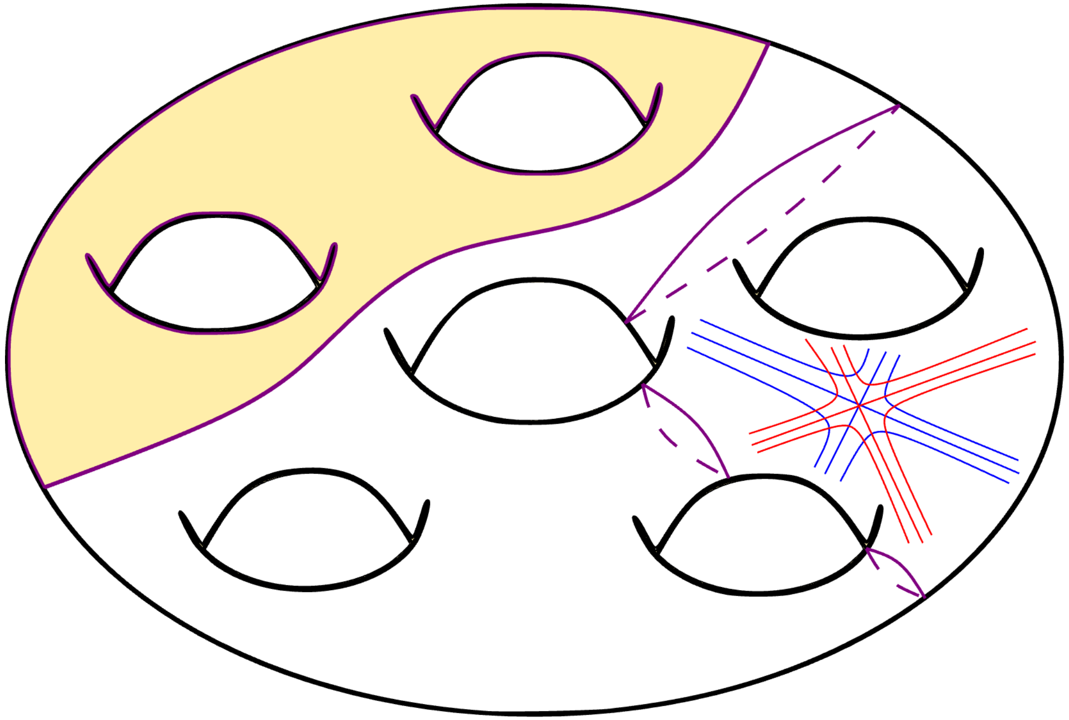
Pseudo-Anosov



Thurston (1976): $\exists$ a number $\lambda > 1$ and a pair of foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ such that $f(\mathscr{F}^u) = \lambda \mathscr{F}^u$ and $f(\mathscr{F}^s) = \lambda^{-1} \mathscr{F}^s$.

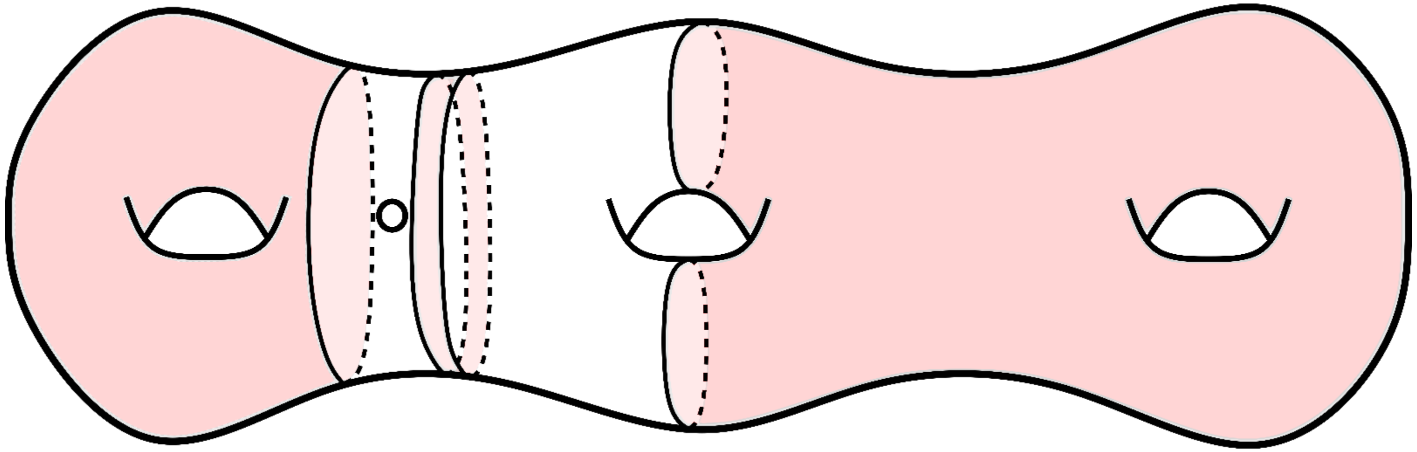
f maps no curve back to itself

Reducible



There is a set of disjoint curves fixed by some power of f

"Jordan Form"



Normal form of $f^k \in \mathrm{MCG}(S)$: Each subsurface is fixed.
Shaded regions are either pseudo-Anosov components or Dehn-twists.
Unshaded regions are fixed.

Pseudo-Anosov: Example

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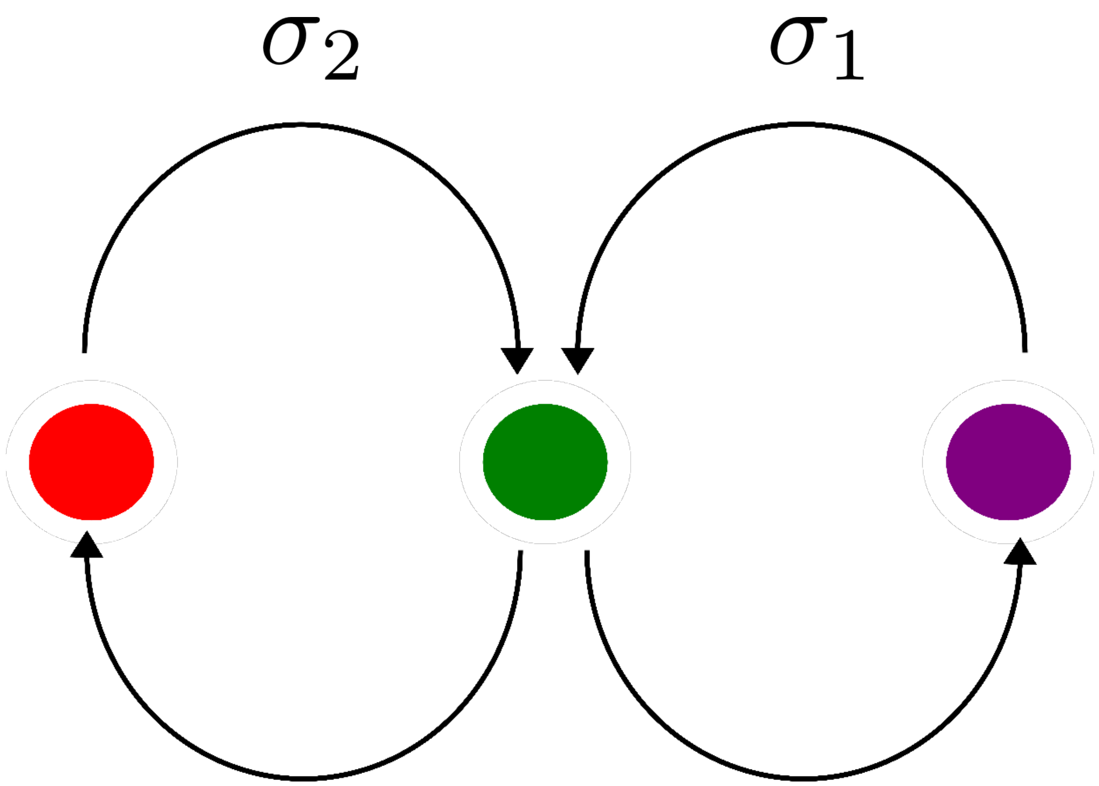

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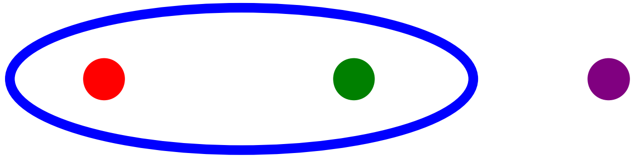

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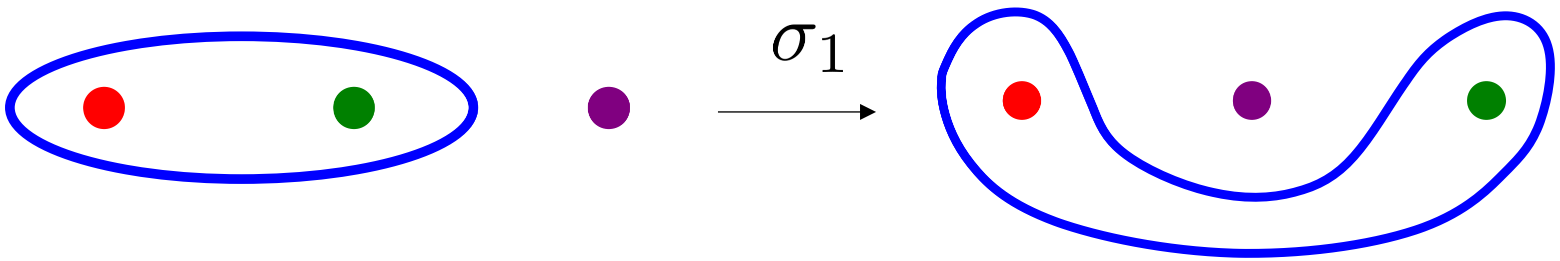

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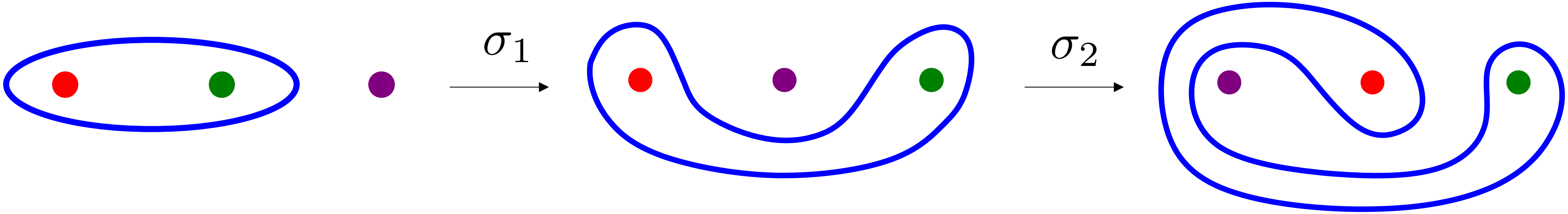

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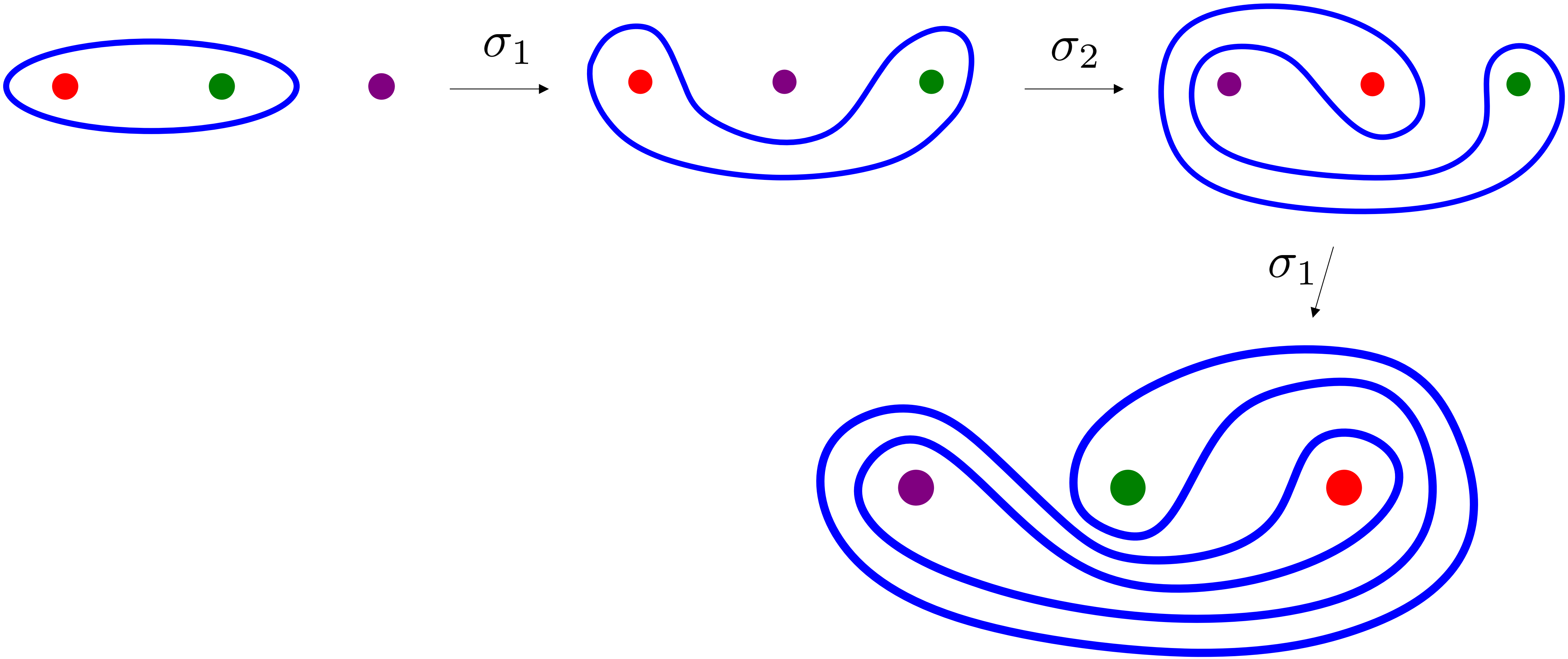

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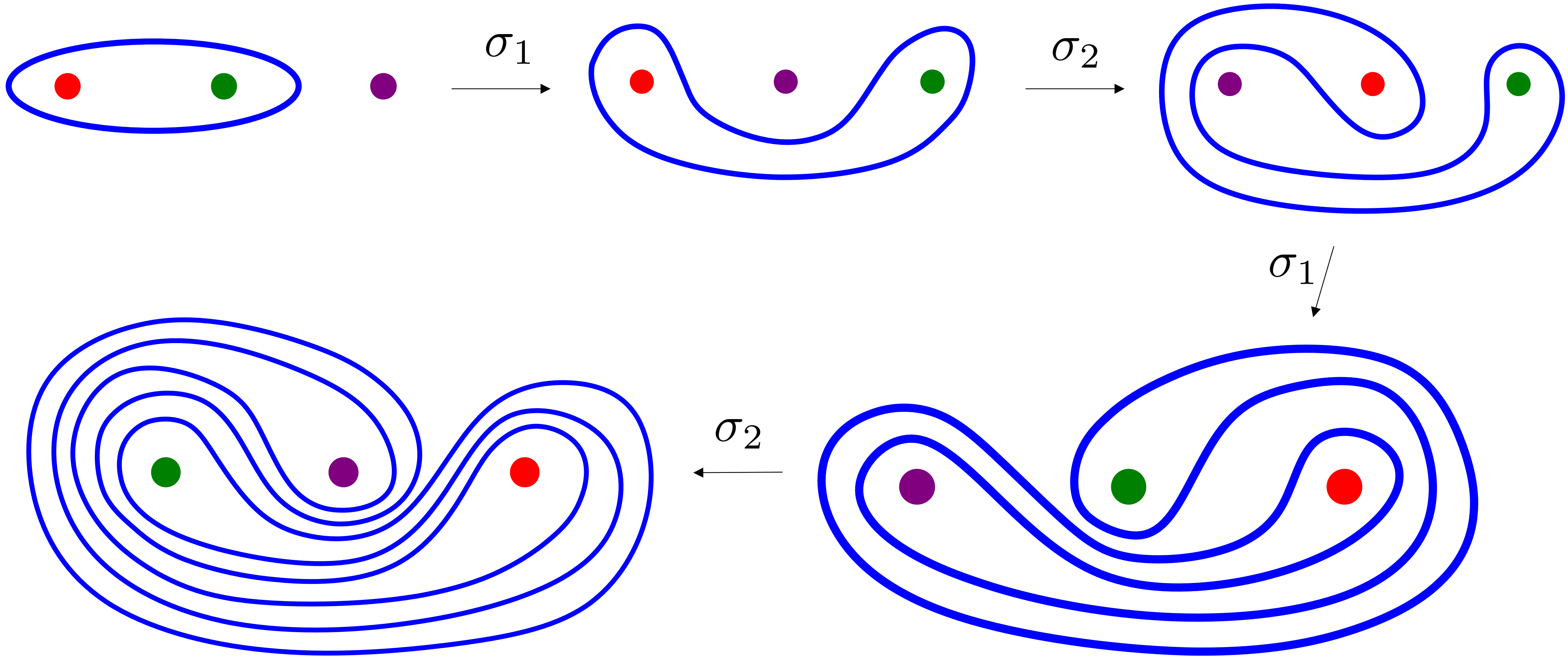

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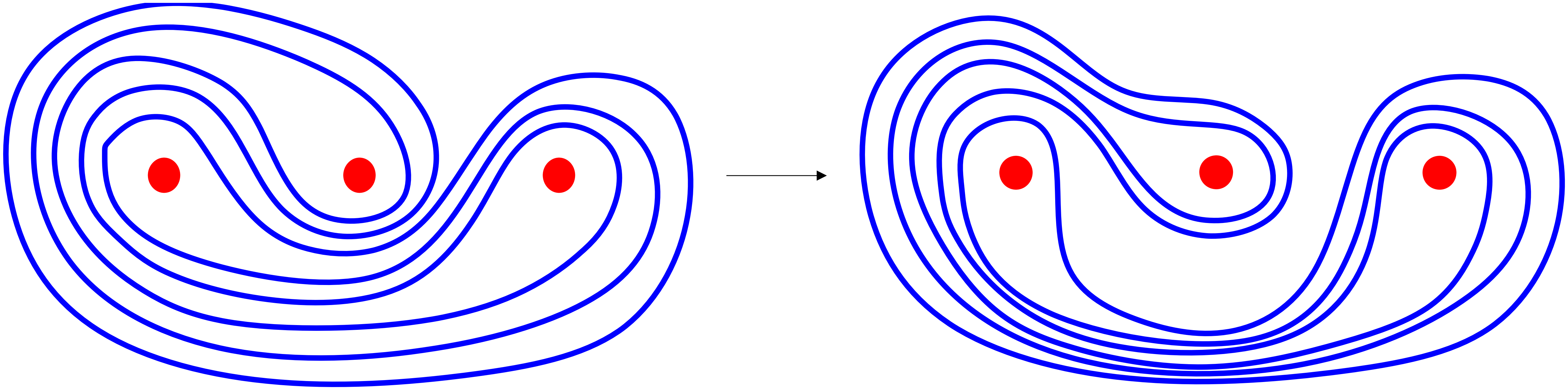

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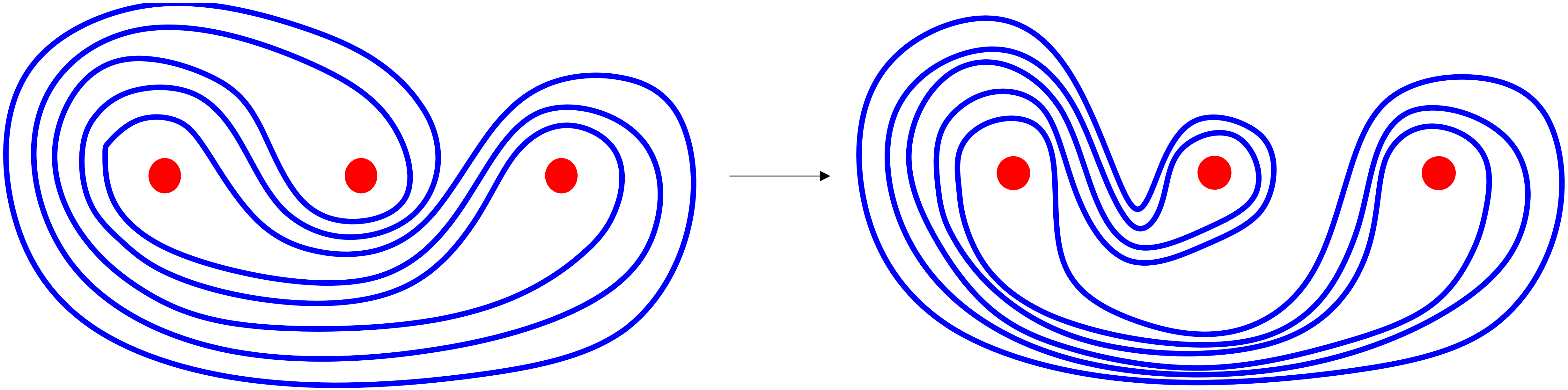

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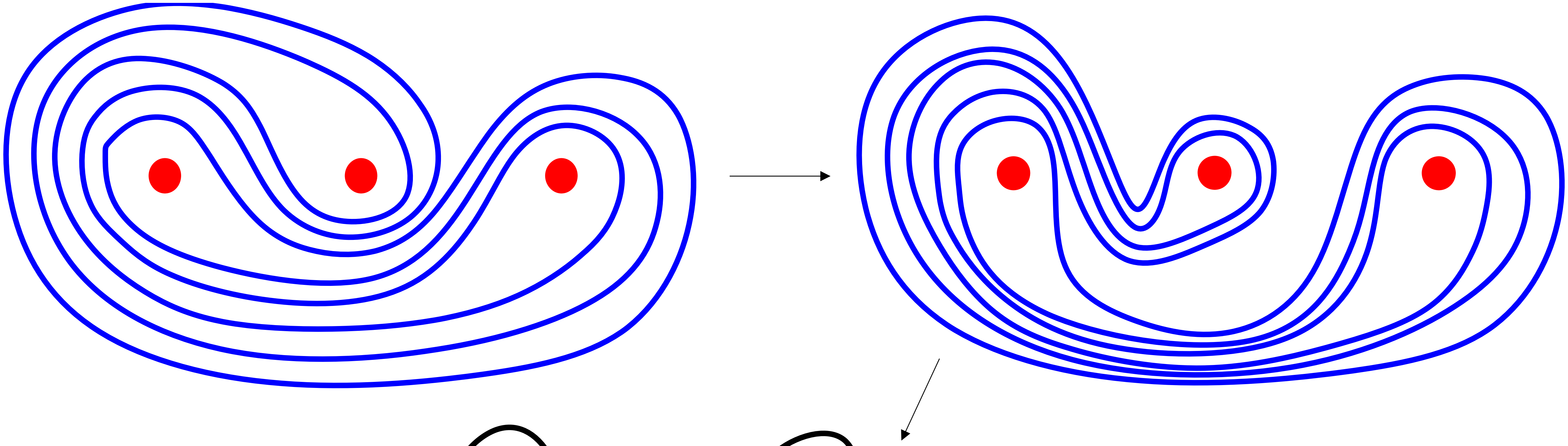
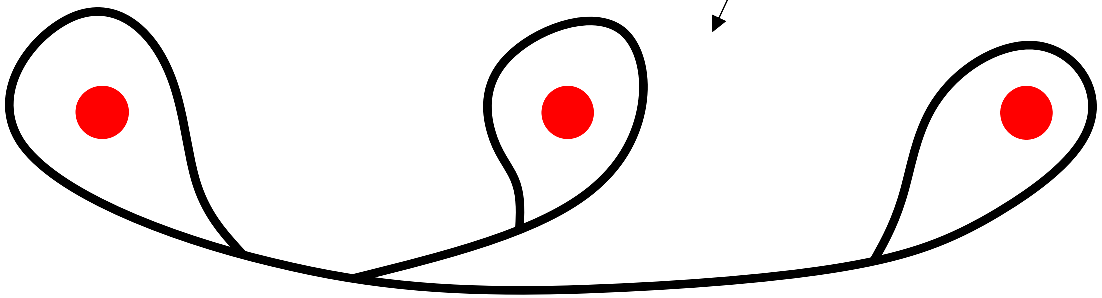


“Train track”



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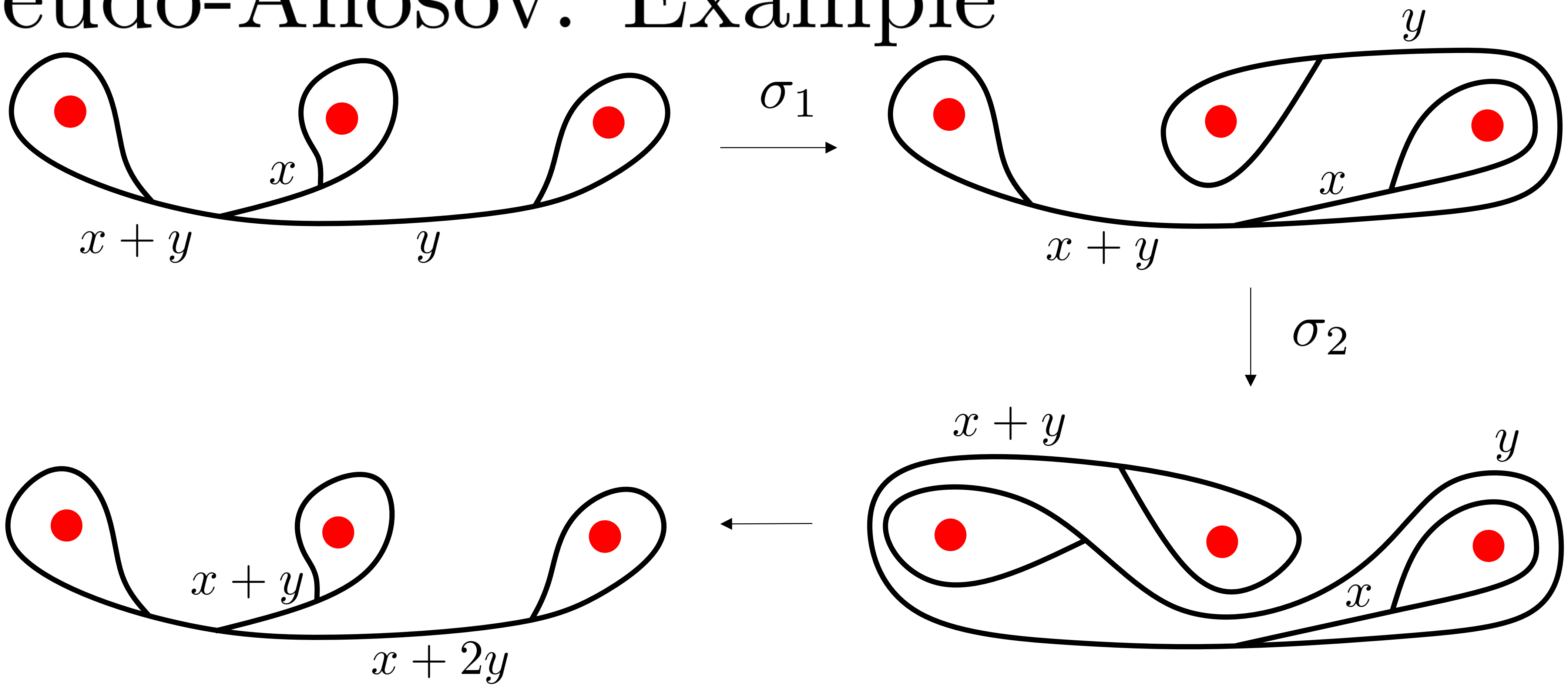

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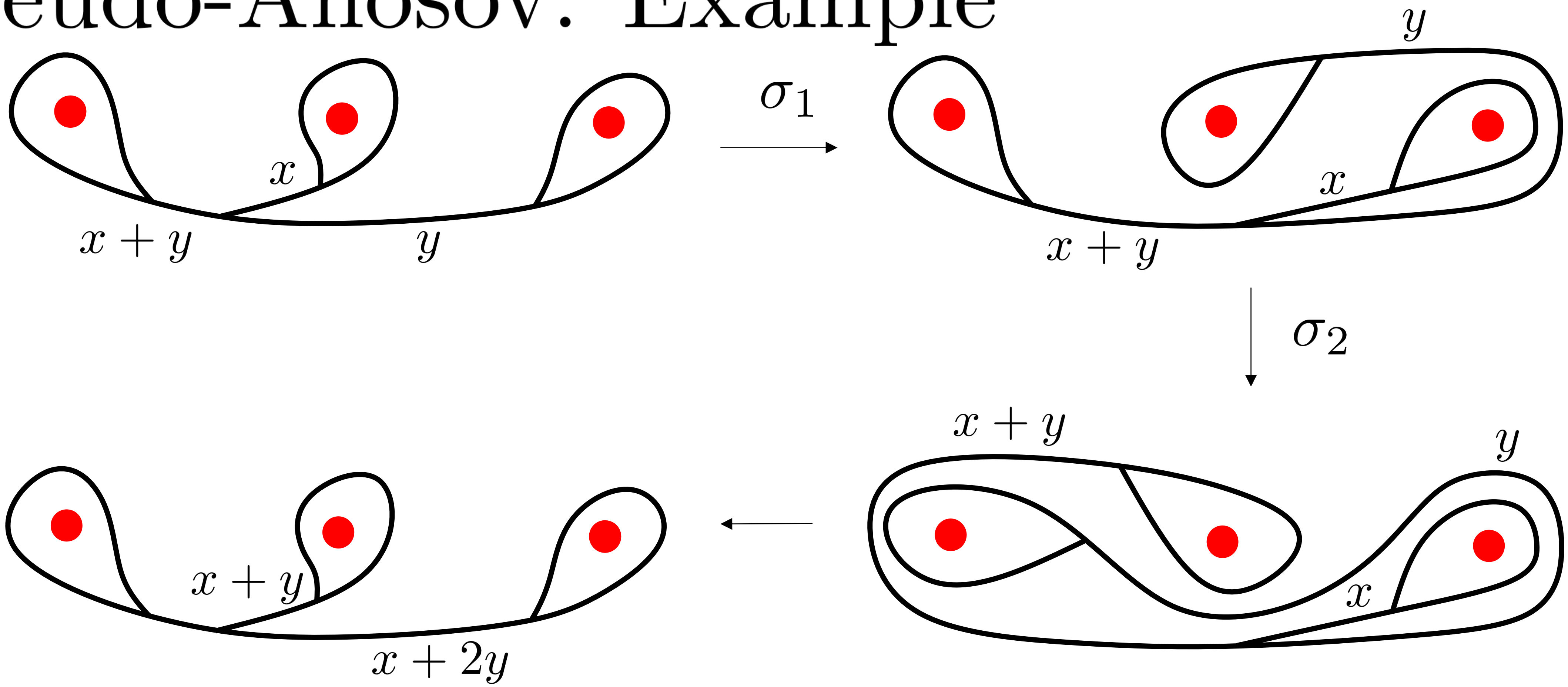

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x + 2y \end{pmatrix}$$

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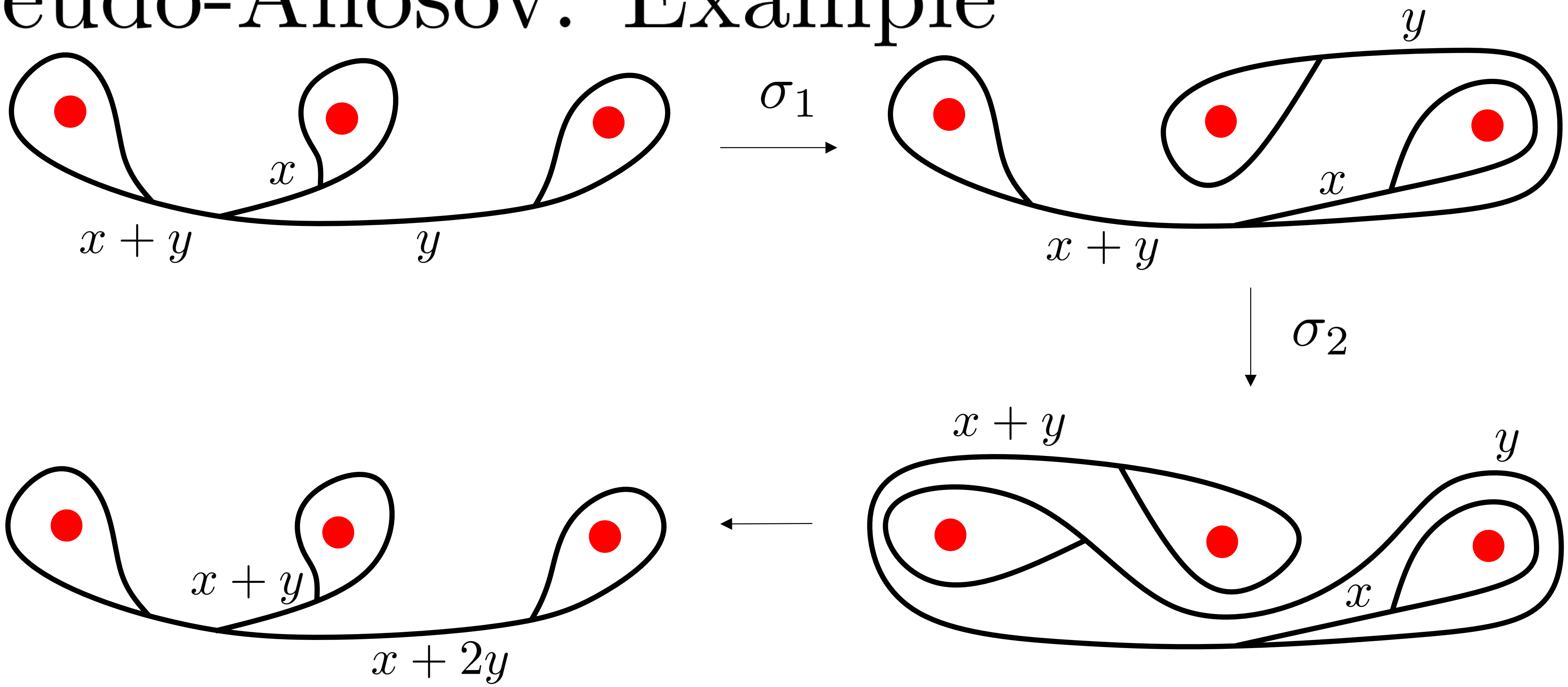

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Eigenvalue = Stretch Factor

Eigenvectors = Foliations

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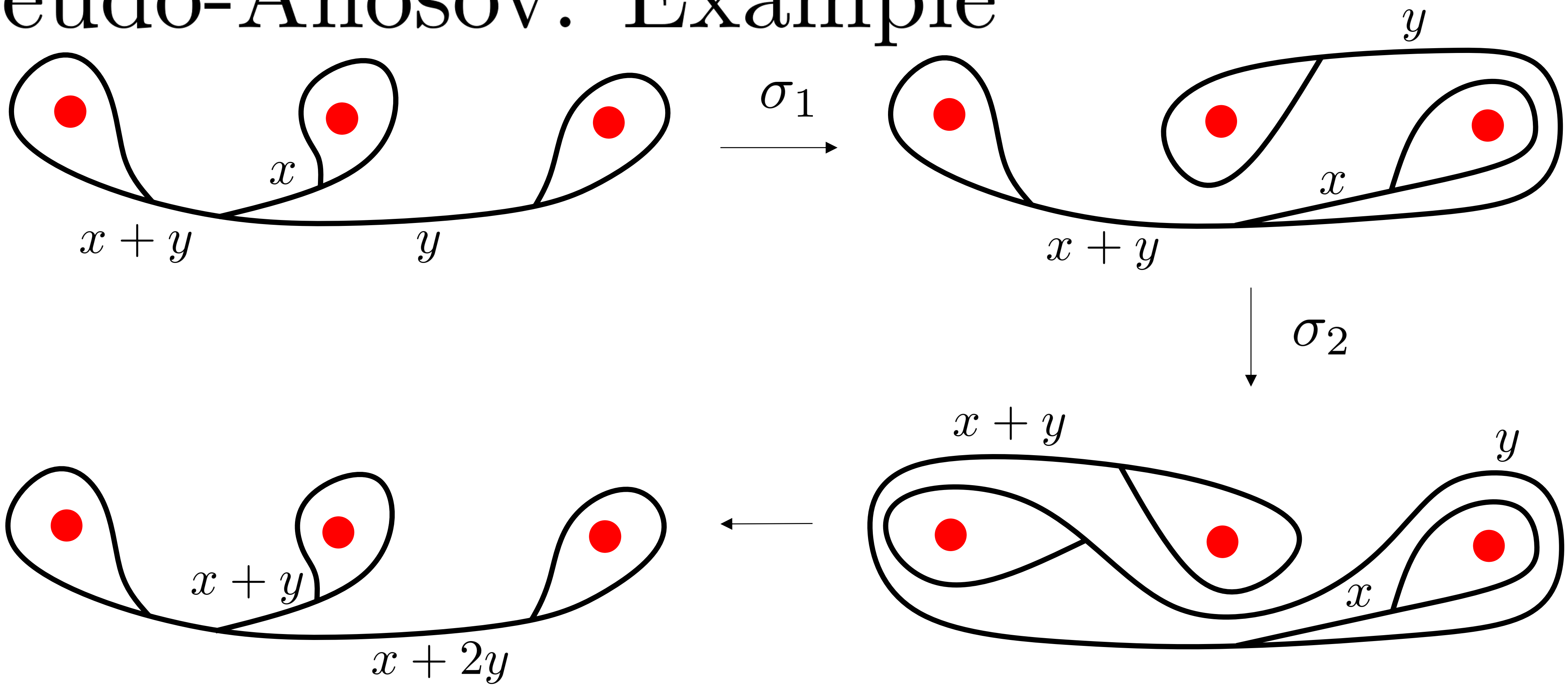

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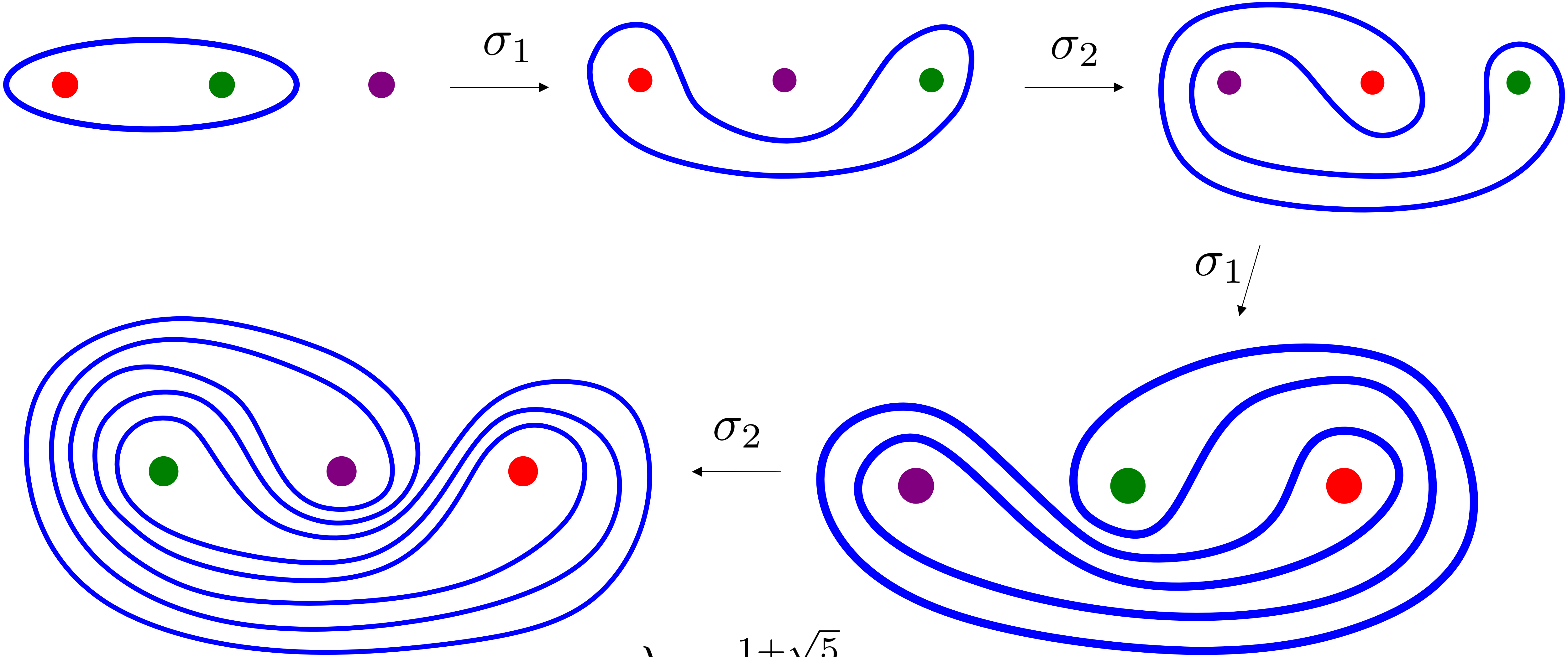

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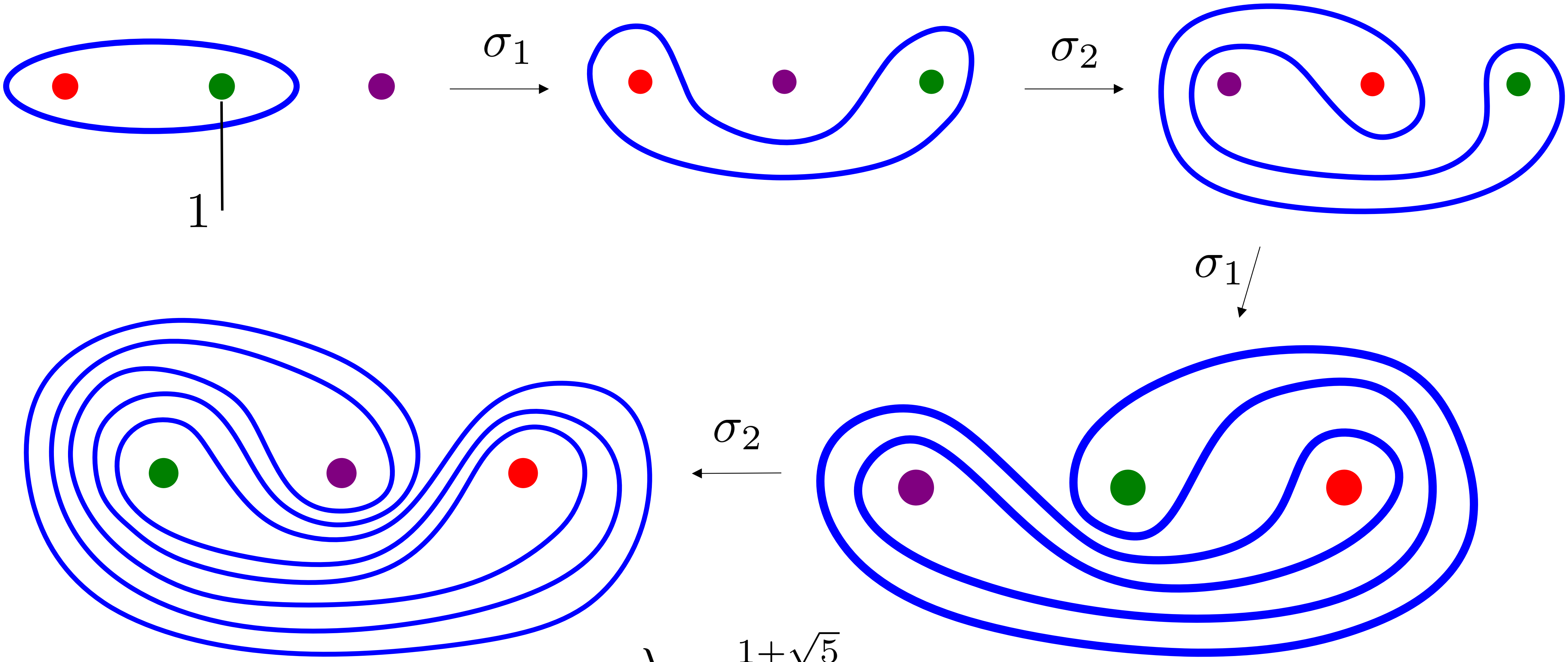

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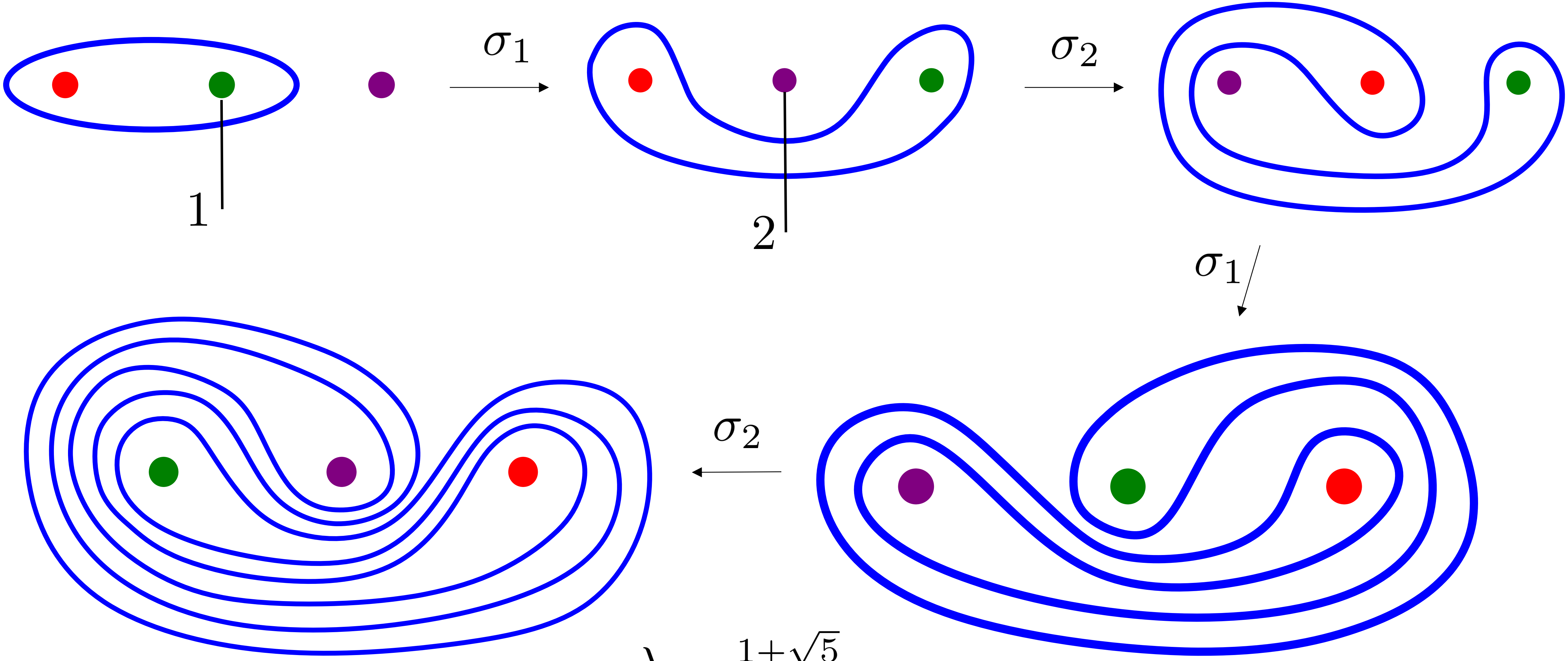

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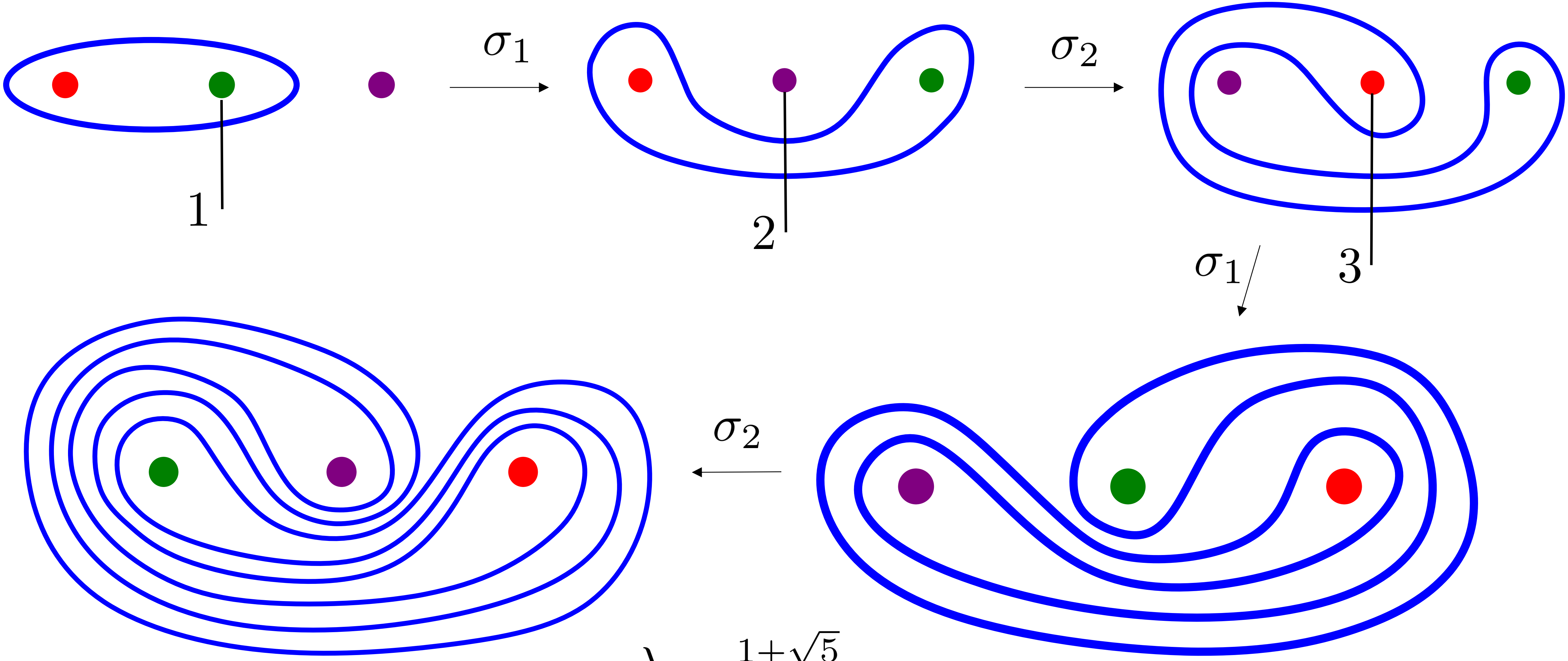

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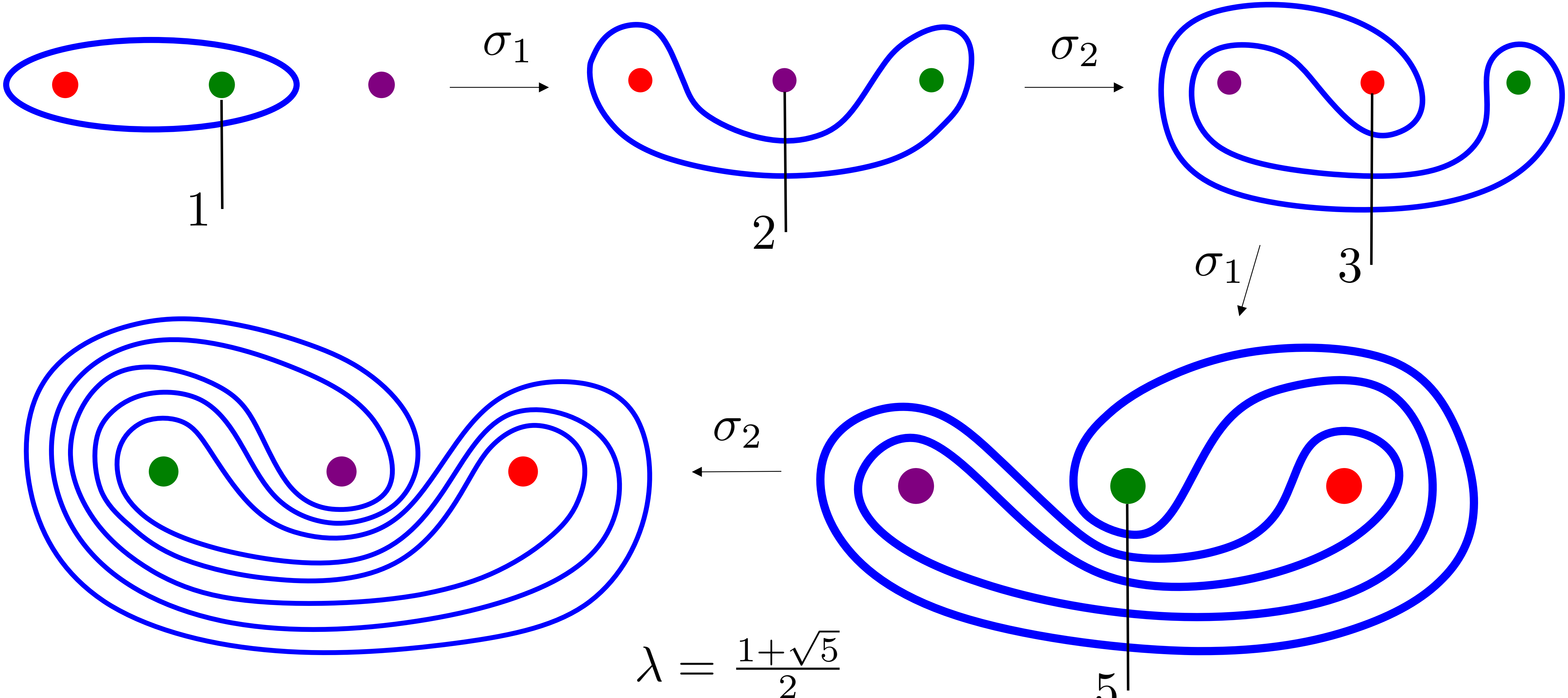

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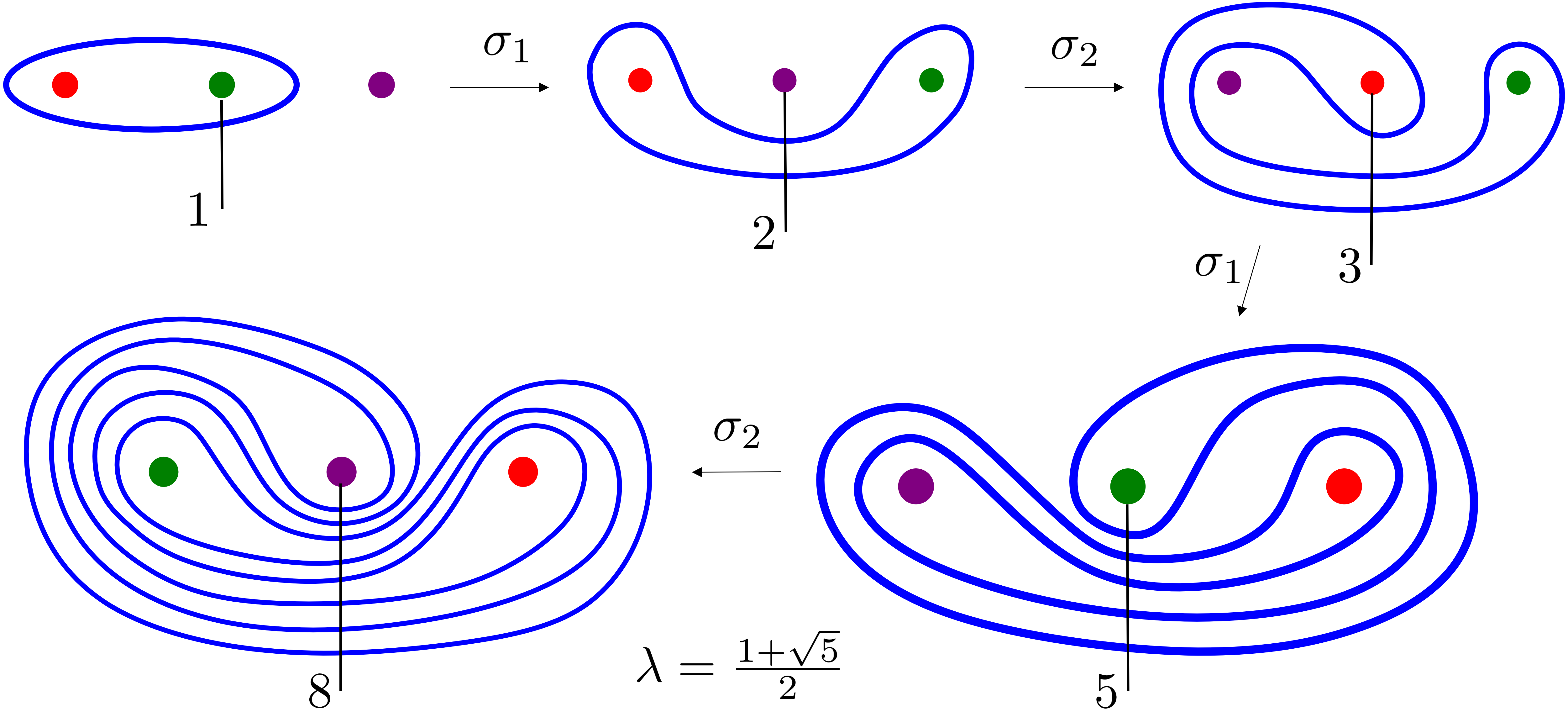
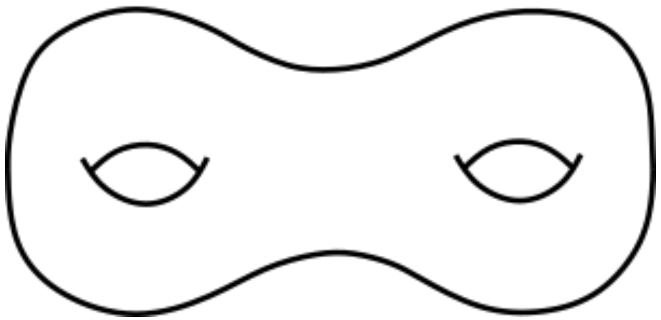


Connection to 3-Manifolds

$$M_f = \frac{S_g \times [0,1]}{(x,0) \sim (f(x),1)}$$



S_2

Connection to 3-Manifolds

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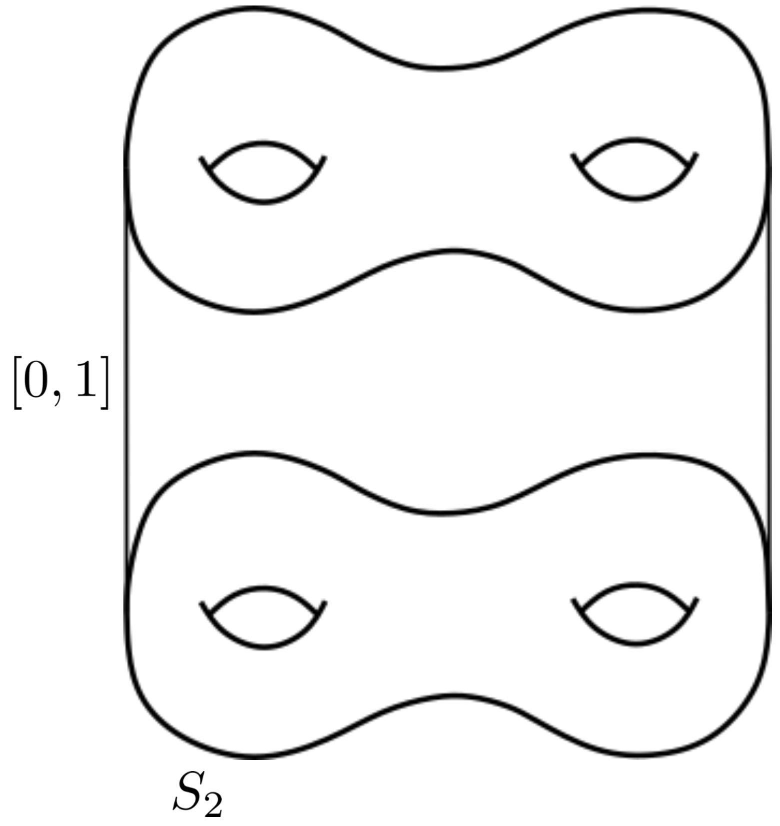

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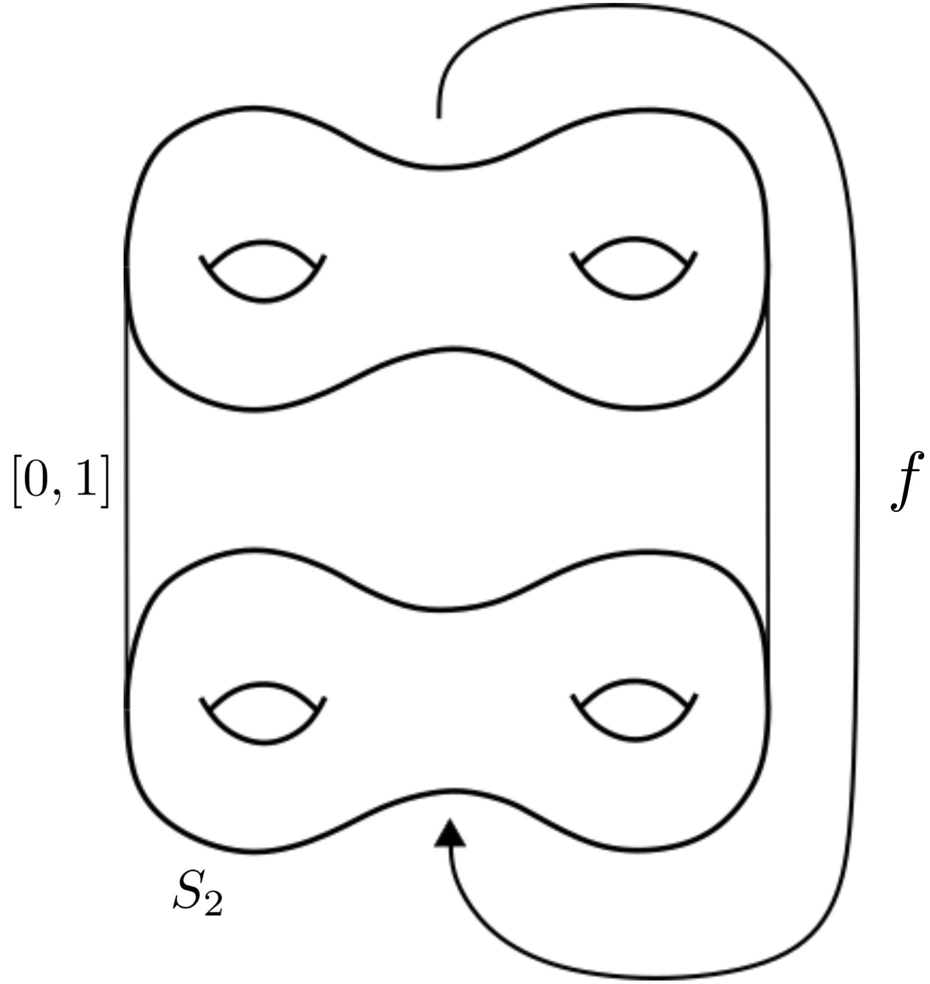

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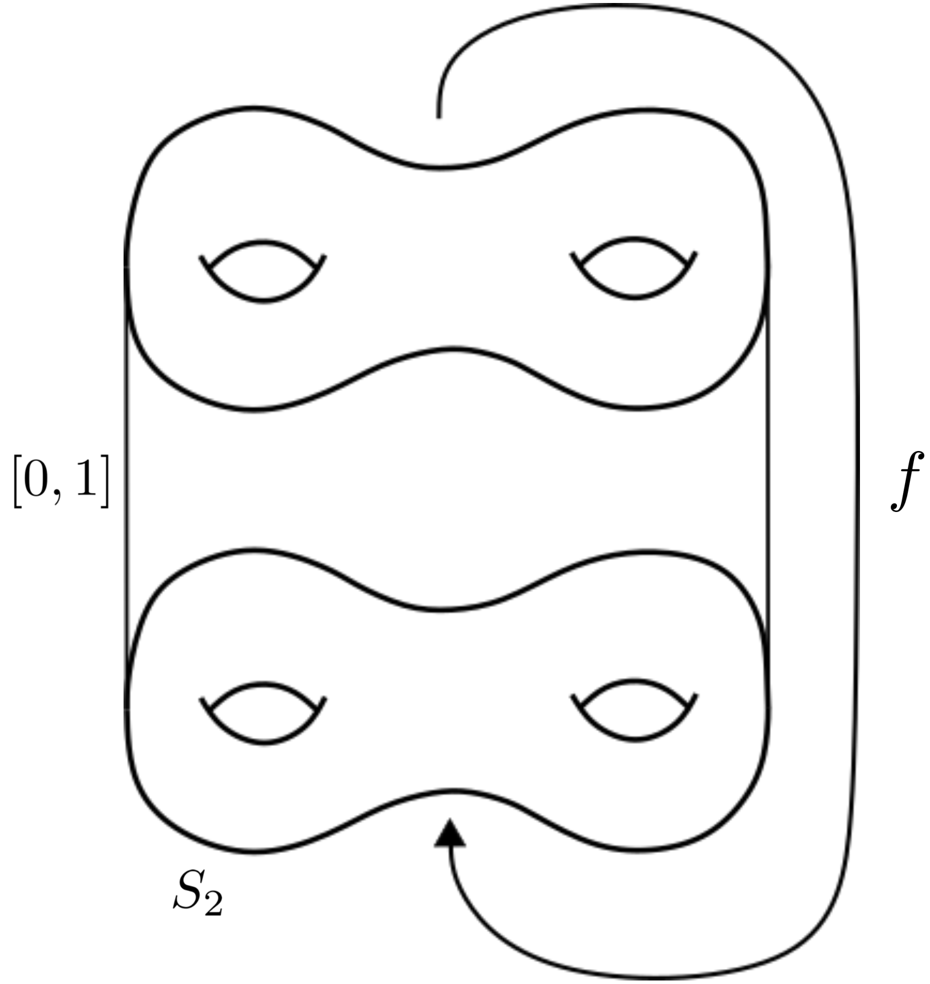


Thurston: $g \geq 2$, then

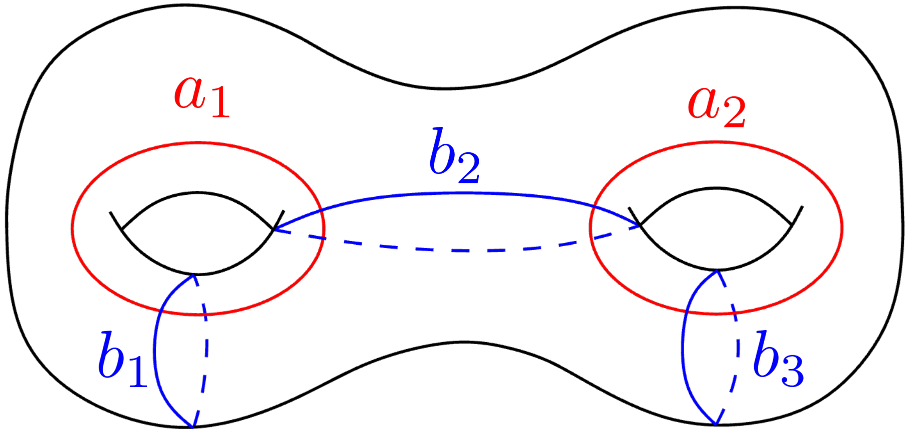
- f periodic $\iff$ M_f admits metric locally isometric to $\mathbb{H}^2 \times \mathbb{R}$
- f reducible $\iff$ M_f contains an incompressible torus
- f pseudo-Anosov $\iff$ M_f admits a hyperbolic metric

Constructing Pseudo-Anosov Maps

A multicurve in S is the union of a finite collection of disjoint simple closed curves in S

A and B are filling multicurves if the complement of $A \cup B$ is a union of disks and once punctured disks

$D_A = \prod_{i=1}^{n} D_{\alpha_i}$ is a multitwist



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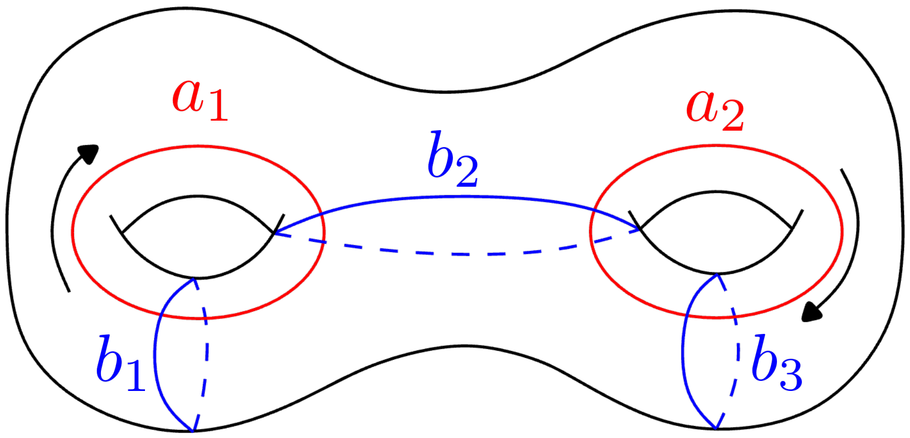

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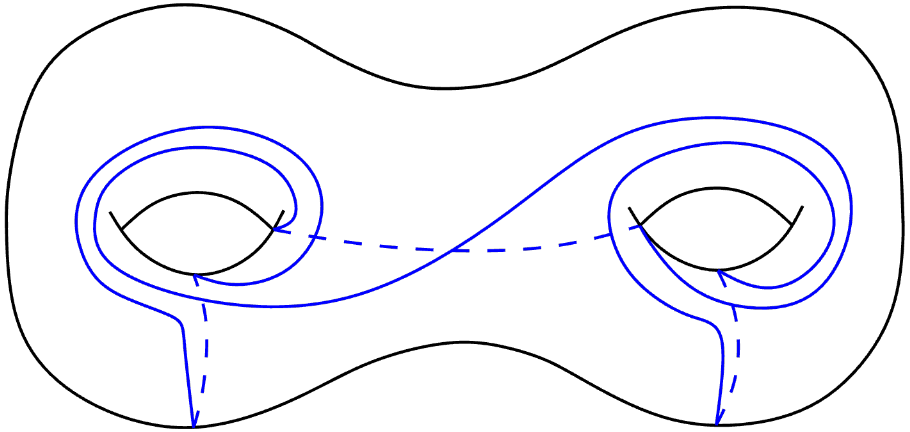

Constructing Pseudo-Anosov Maps

Thurston: A and B are filling multicurves in S. There is a representation $\rho : \langle D_A, D_B \rangle \longrightarrow PSL(2, \mathbb{R})$ such that $f \in \langle D_A, D_B \rangle$ is pseudo-Anosov if and only if $|Tr(\rho(f))| > 2$.

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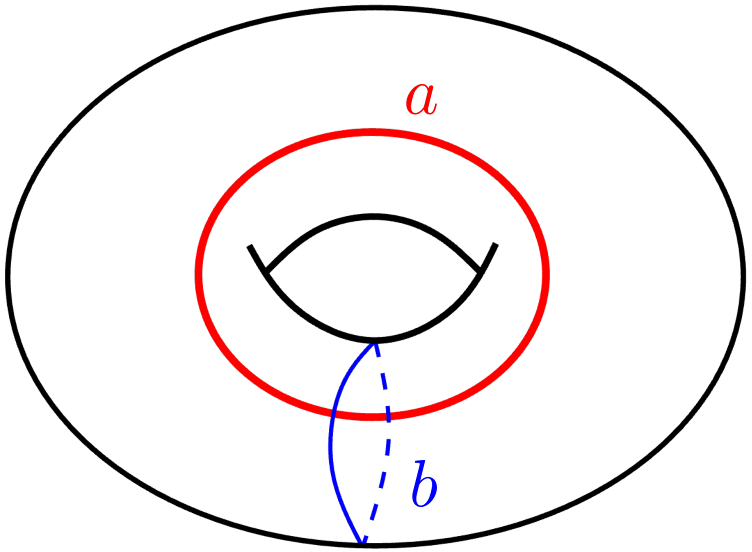
$\mathrm{MCG}(S_{1,0}) \cong SL(2,\mathbb{Z})$

$f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$fg = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$



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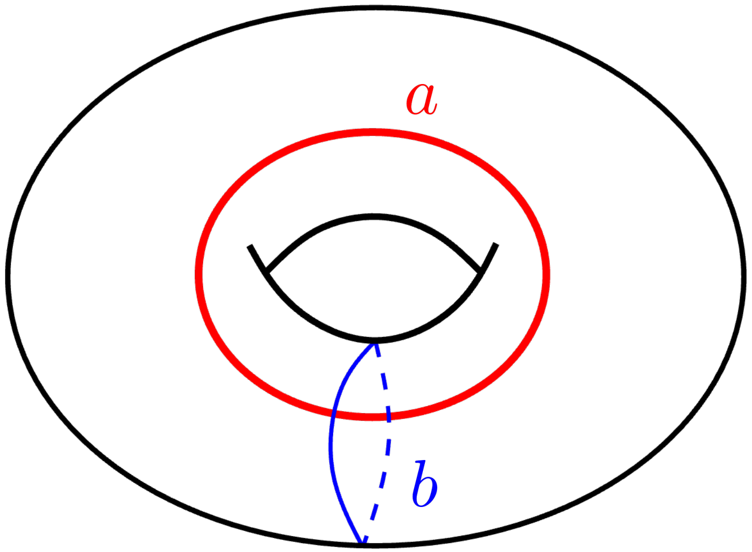


Thurston's construction $\implies$ fg pseudo-Anosov

ie. positive twist around a followed by negative twist around b is pseudo-Anosov

Constructing Pseudo-Anosov Maps

Penner: Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ be filling multicurves on S. Then any product of positive Dehn twists about a_j and negative Dehn about b_k is pseudo–Anosov provided that all $n + m$ Dehn twists appear in the product at least once.

Constructing Pseudo-Anosov Maps

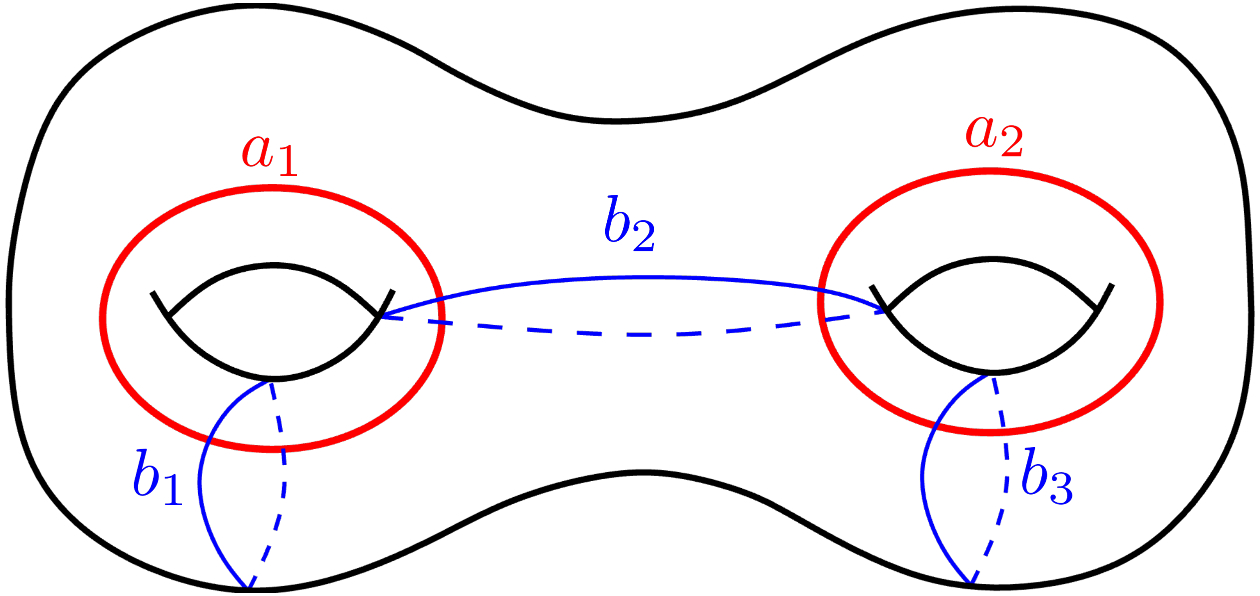
Penner: Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ be filling multicurves on S. Then any product of positive Dehn twists about a_j and negative Dehn about b_k is pseudo–Anosov provided that all $n + m$ Dehn twists appear in the product at least once.

$A = \{a_1, a_2\}$

$B = \{b_1, b_2, b_3\}$

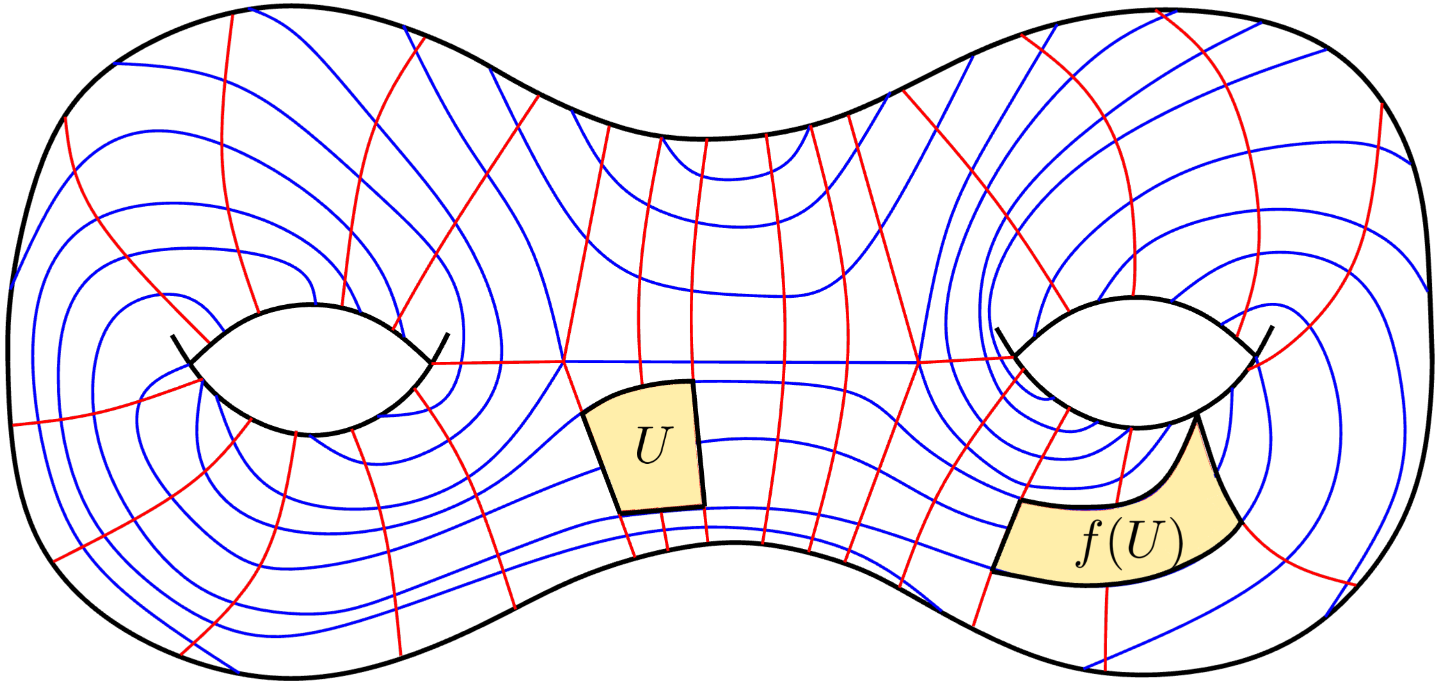
Positive Dehn twists around curves in A

Negative Dehn twists around curves in B



Penner's construction $\implies$ pseudo-Anosov

Pseudo-Anosov



f maps no curve back to itself

Thurston (1976): $\exists$ a number $\lambda > 1$ and a pair of foliations $\mathscr{F}^u$ and $\mathscr{F}^s$ such that $f(\mathscr{F}^u) = \lambda \mathscr{F}^u$ and $f(\mathscr{F}^s) = \lambda^{-1} \mathscr{F}^s$.

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Theorem (V.): No.

Main difficulty: How can you tell two pseudo-Anosov homeomorphisms are different?

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Tool: Use number theoretic properties associated to the stretch factor.

Theorem (V.): There exists a construction of pseudo-Anosov mapping classes which produce new pseudo-Anosovs unique from Thurston and Penner.

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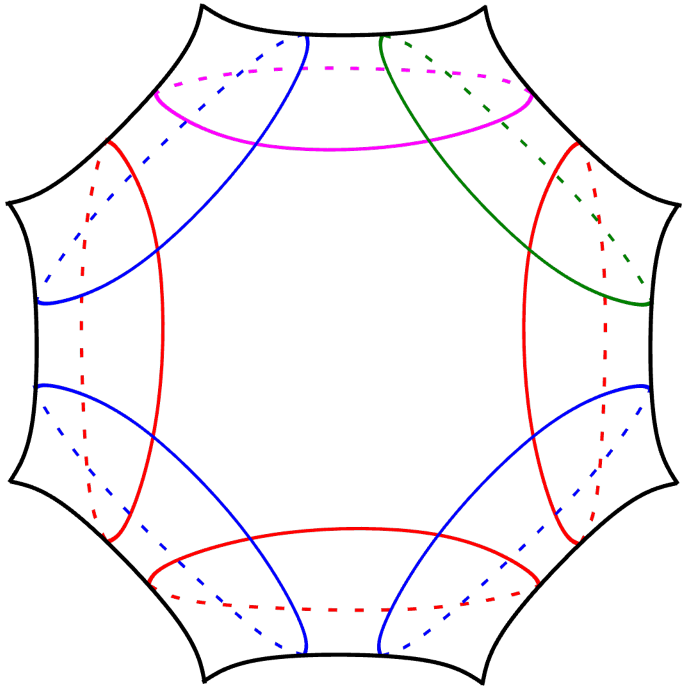

Twist red curves
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Twist magenta curve
Twist green curve
$\rightsquigarrow$ pseudo-Anosov map

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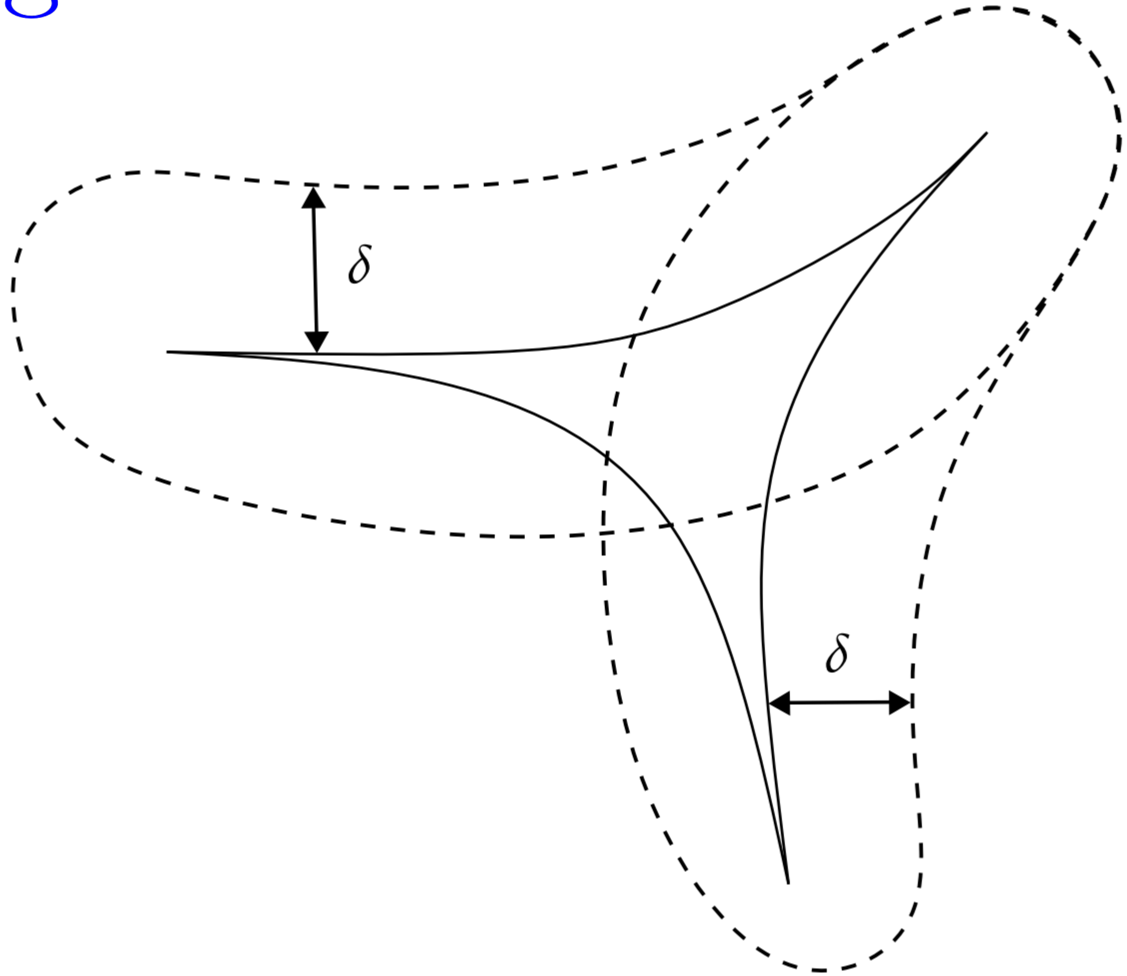
Thin Triangles Condition:

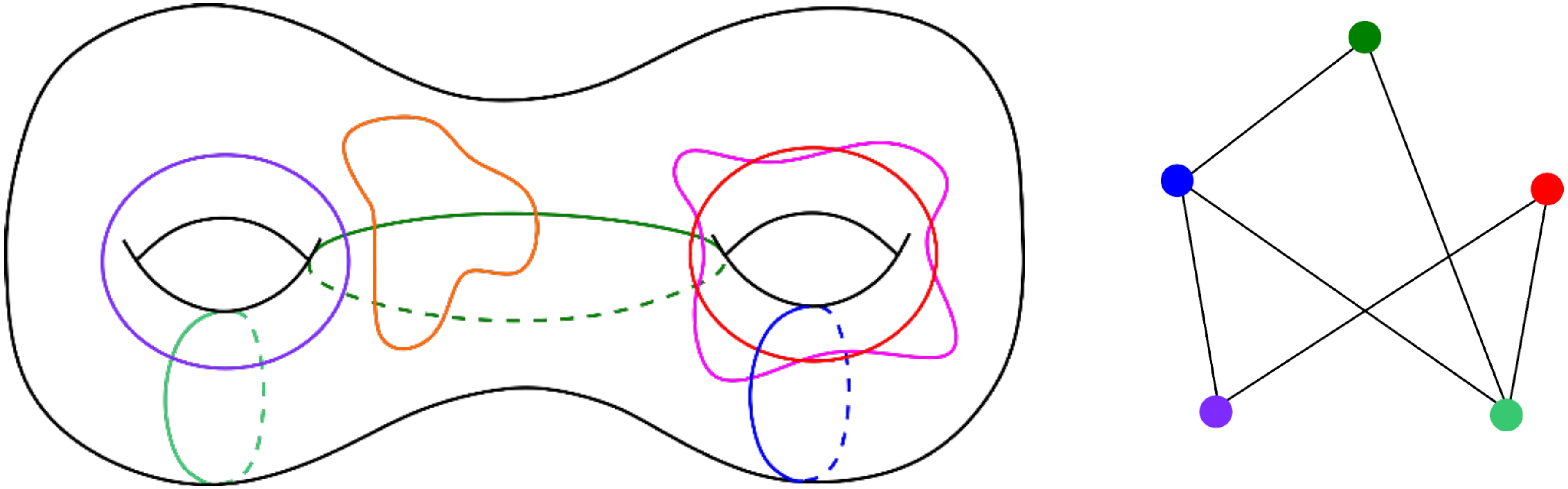


A geodesic metric space is Gromov hyperbolic if it satisfies the thin triangle condition.

Curve Graph (Harvey [1988])

Vertices: Homotopy classes of essential simple closed curves

Edges: Disjointness



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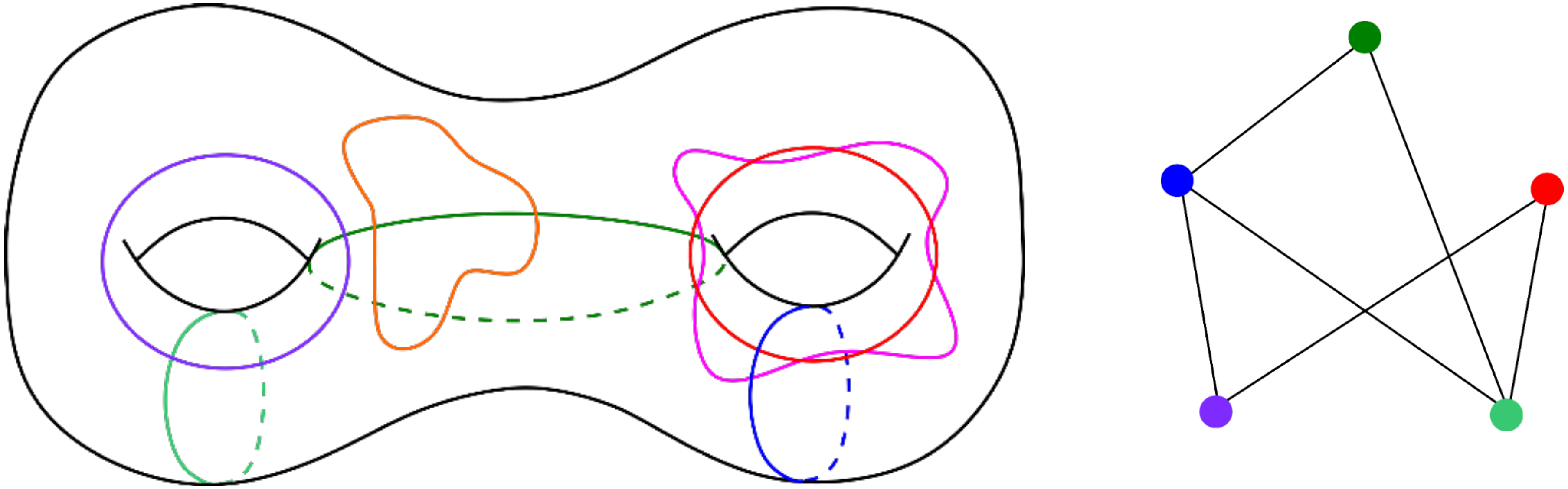

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Ivanov(1997): For $g \geq 3$, the natural map

$$\mathrm{MCG}(S_g) \to \mathrm{Aut}(C(S_g))$$

is an isomorphism.

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$\rightsquigarrow \mathcal{C}(S)$ a combinatorial tool to study $\mathrm{MCG}(S)$

$\mathrm{MCG}(S) \curvearrowright \mathcal{C}(S)$

Masur–Minsky(1999): $f \in \mathrm{MCG}(S)$ acts on $\mathcal{C}(S)$:

- elliptic if every orbit of f is bounded

  i.e. periodic and reducible

- hyperbolic if f translates along an axis.

  i.e. pseudo-Anosov

Consequence: The curve graph is infinite diameter.

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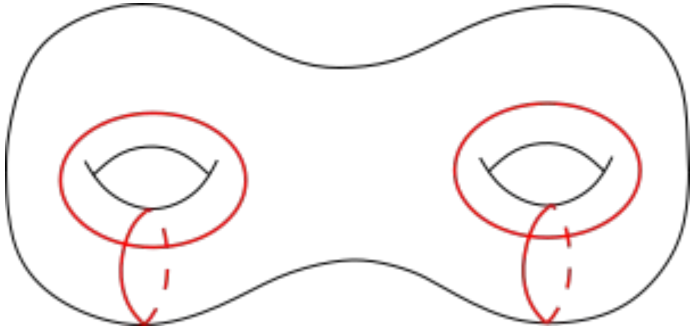
Mapping class groups

S is finite-type if the fundamental group is finitely generated



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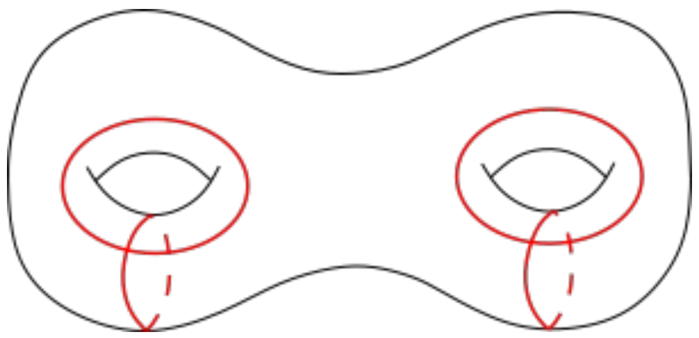

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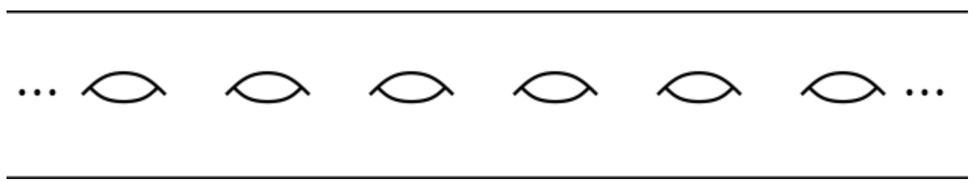

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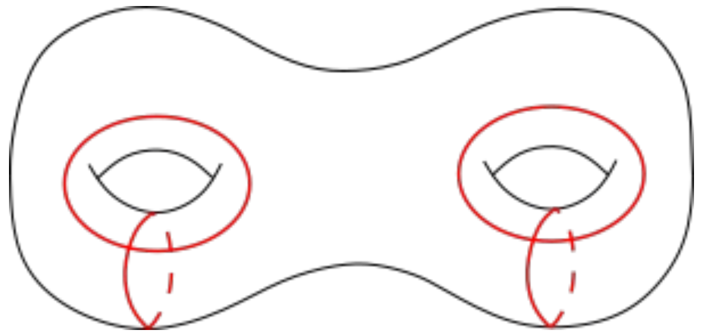

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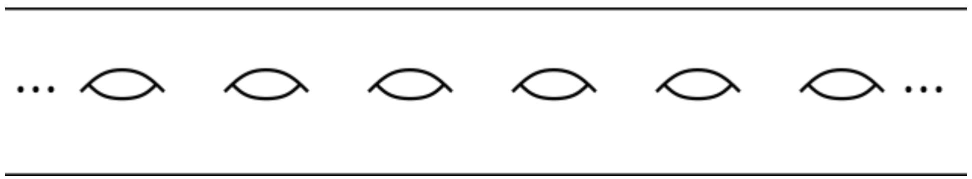

Mapping class groups of infinite type surfaces are called big mapping class groups

Why study infinite type surfaces?

- Connections to complex dynamics

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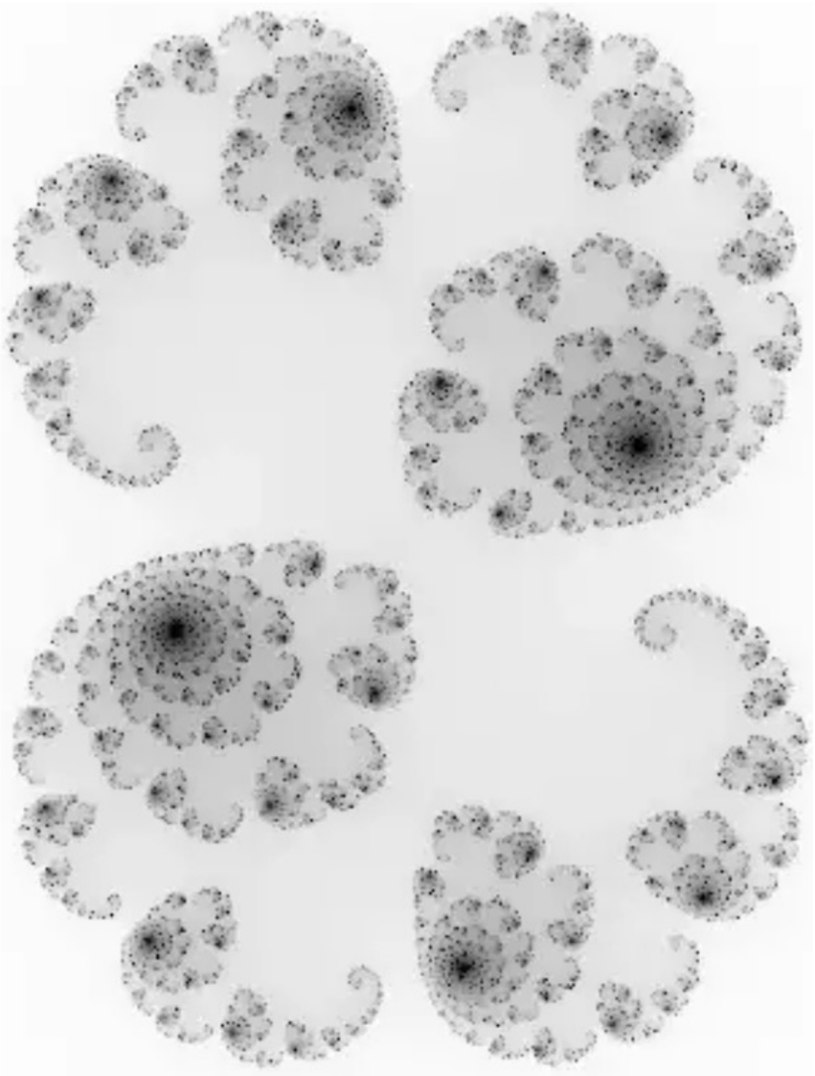

Julia set when $c = 0.285 + 0.01i$

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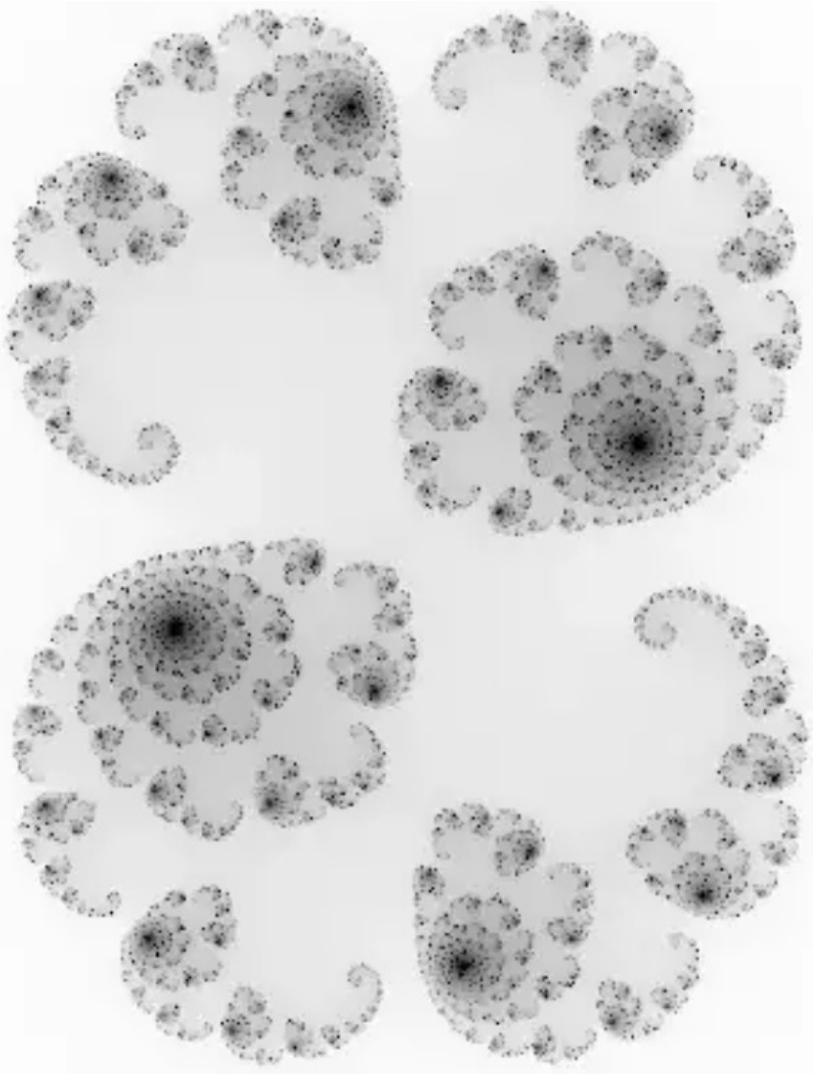
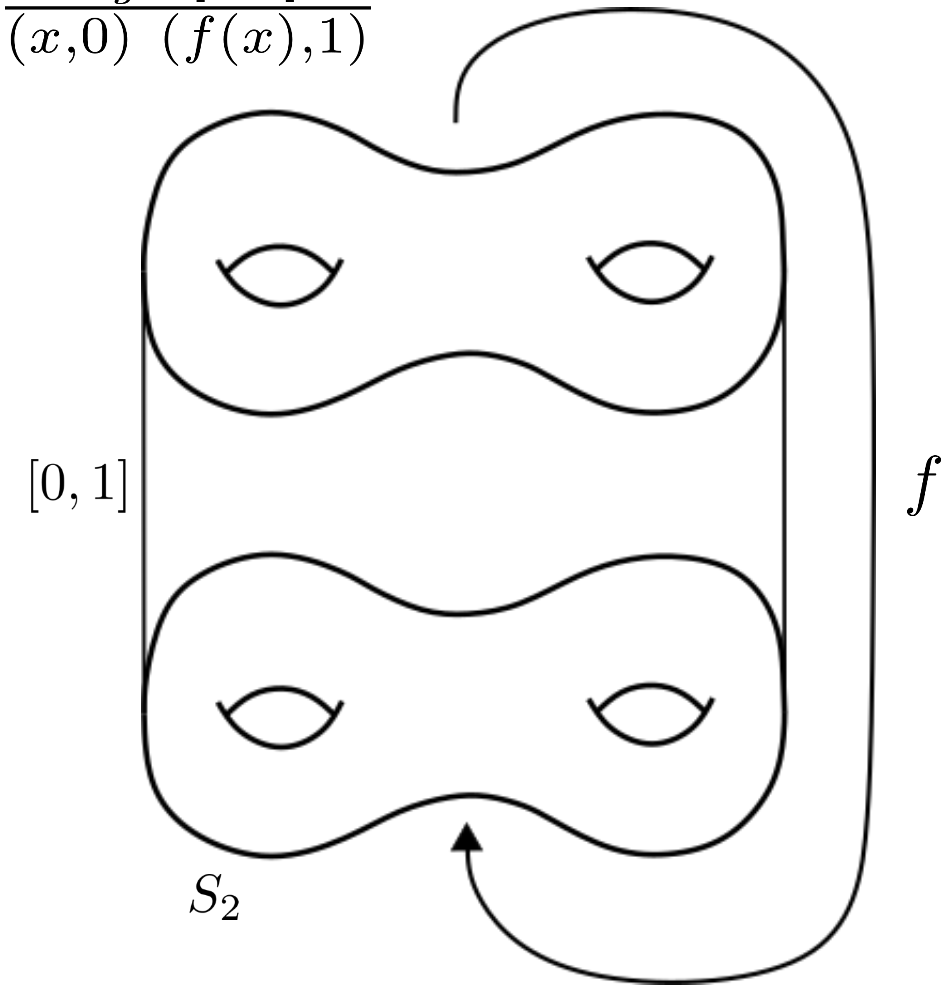
Vary the parameter $c \in \mathbb{C}$



Julia set when $c = 0.285 + 0.01i$

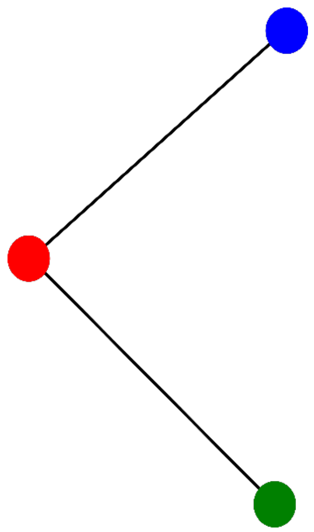
Recall: Connection to 3-Manifolds

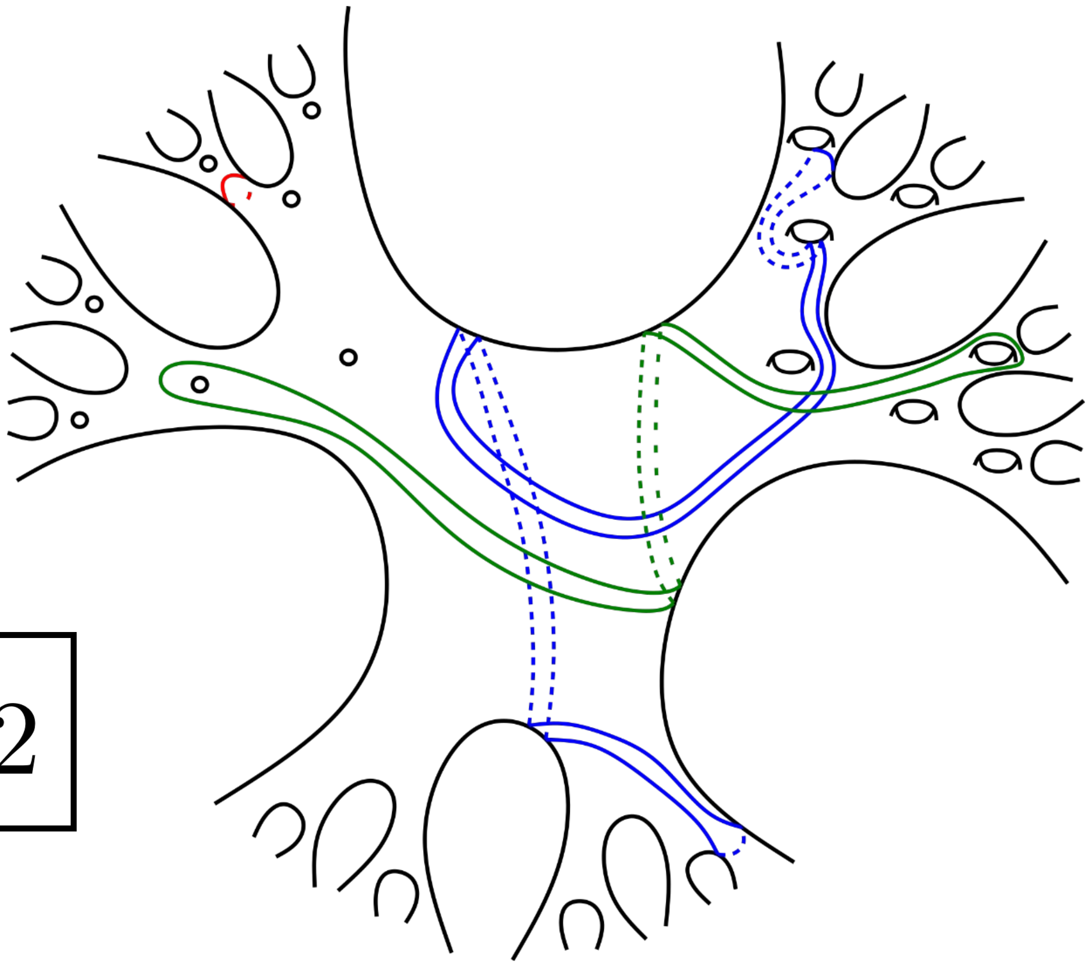
$$M_f = \frac{S_g \times [0,1]}{(x,0) \quad (f(x),1)}$$



Irrational foliation of M_f $\rightsquigarrow$ Infinite type surface

What about infinite type surfaces?





$$\operatorname{diam}(\mathcal{C}(\Sigma)) = 2$$

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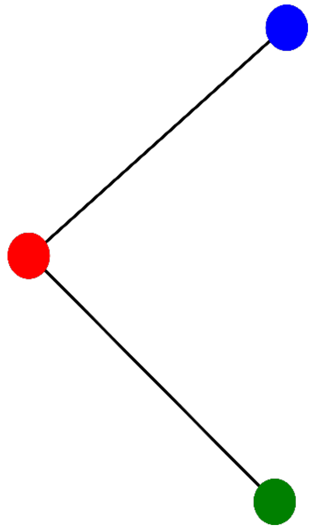

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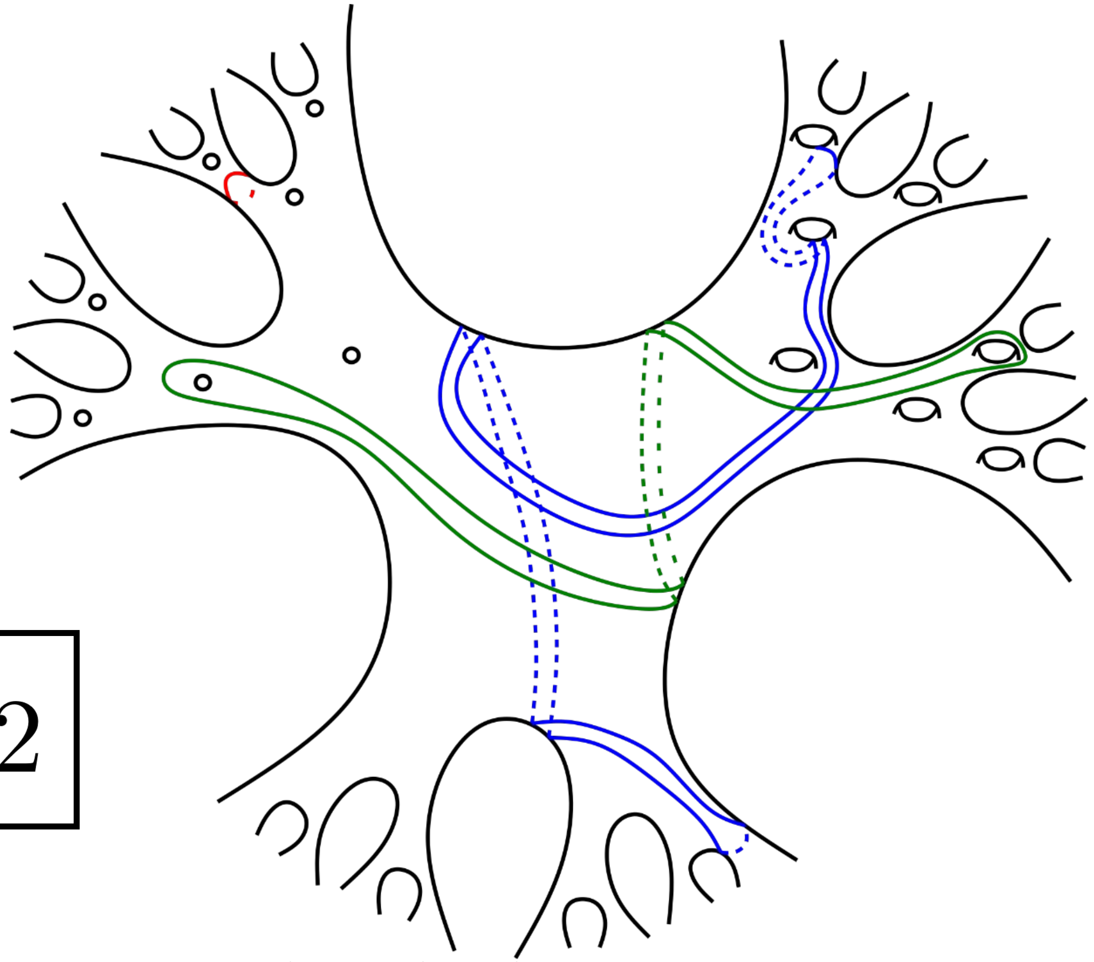
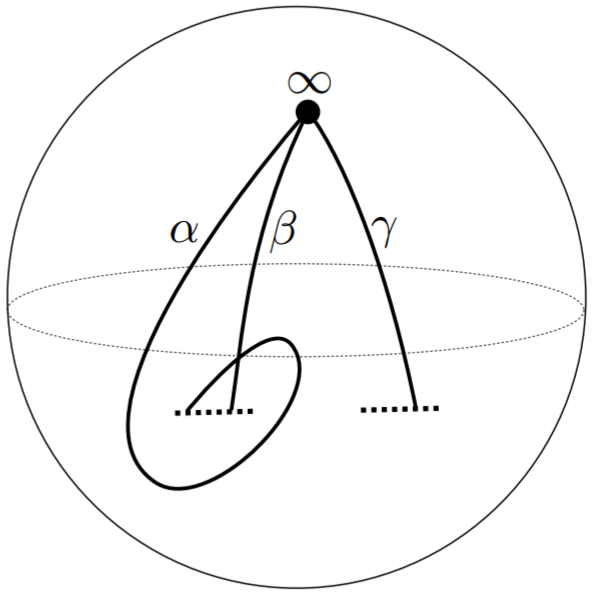


Can we find a graph on which big mapping class groups admit a hyperbolic action?

Ray Graph (Calegari)

Vertices: Isotopy classes of proper rays, with interior in the complement of K, from a point in K to infinity

Edges: Disjointness



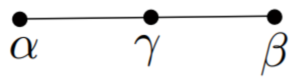


Image by J. Bavard

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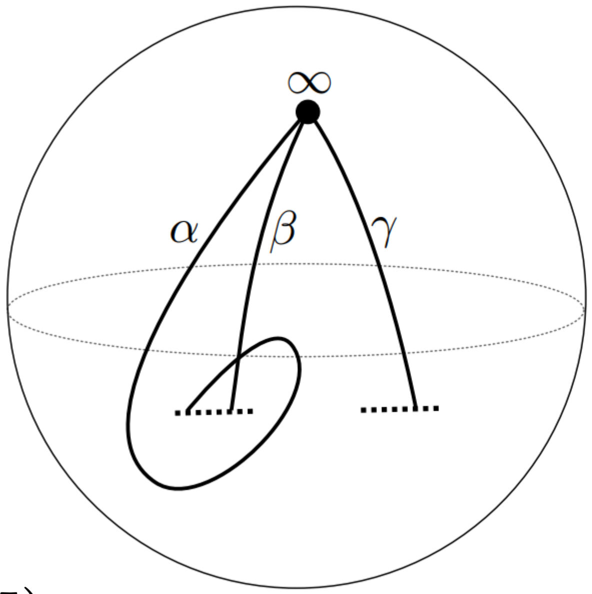
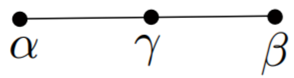


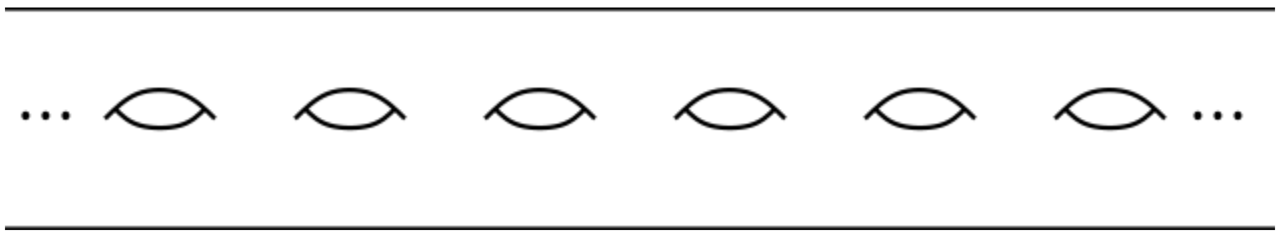


Theorem (Bavard): The ray graph has infinite diameter, is Gromov hyperbolic, and there exists an element of $\mathrm{MCG}(\mathbb{R}^2 \setminus K)$ which acts by translation on a geodesic axis of the ray graph.

Image by J. Bavard

Ends

An end is a way of exiting every compact set of the surface.



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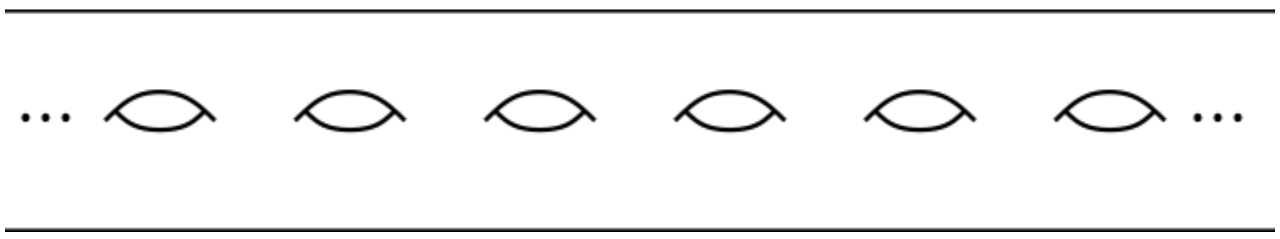

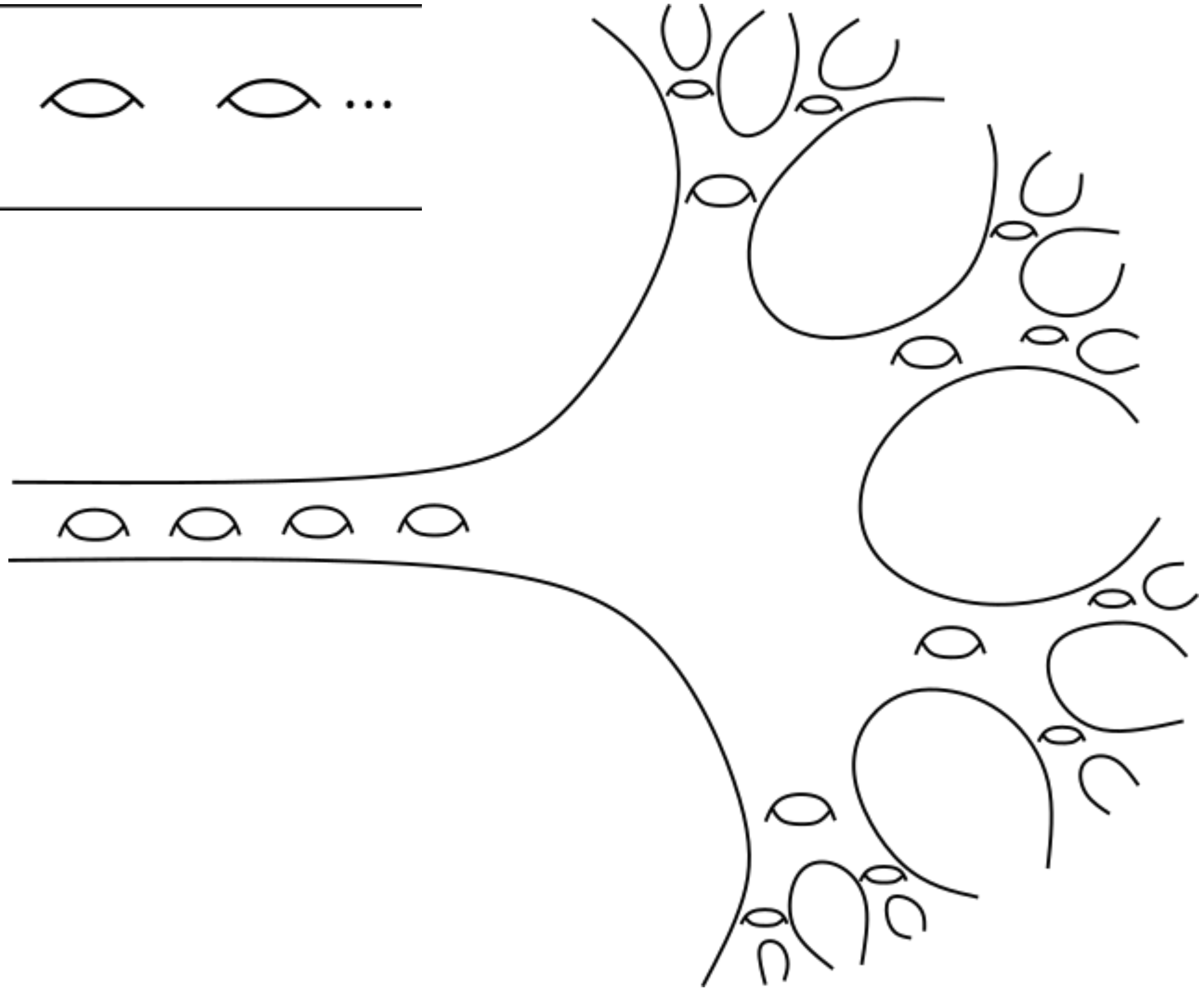

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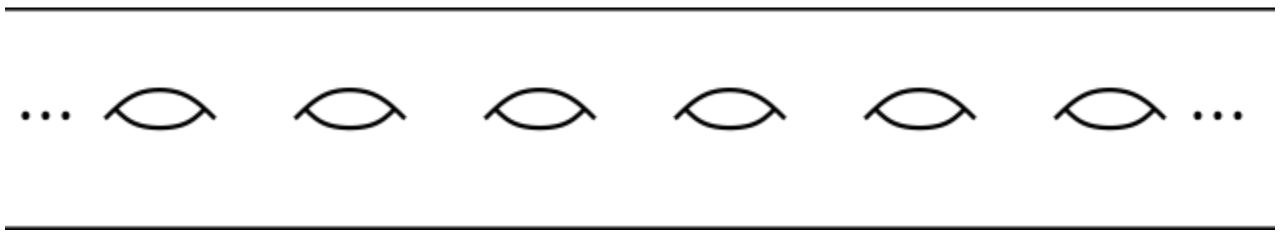

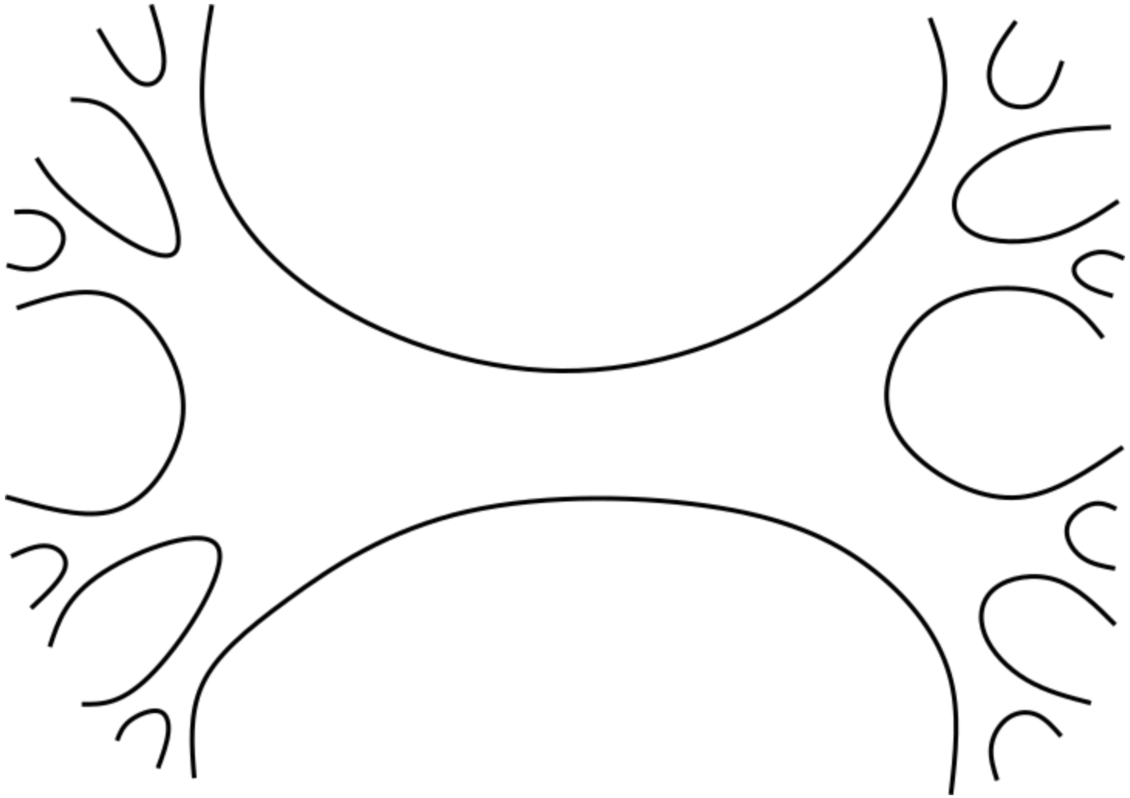

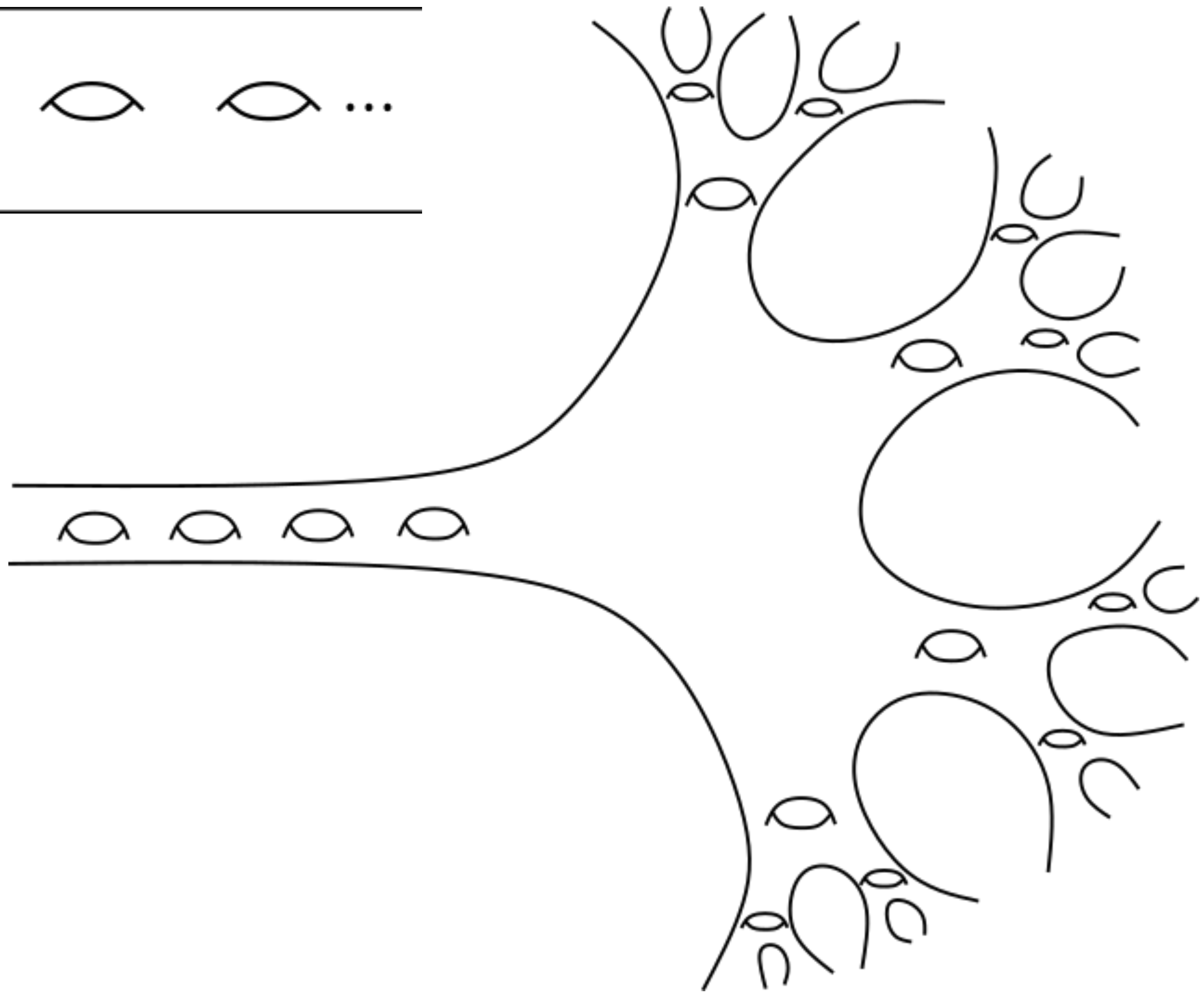

Omnipresent Arc Graph (Fanoni–Ghaswala–McLeay)

One-cut subsurface: complementary component of a separating loop.

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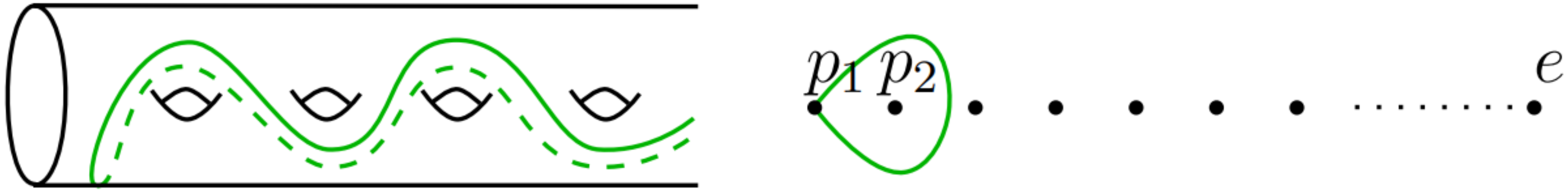
One-cut homeomorphic subsurface: A one-cut subsurface which is homeomorphic to the full surface



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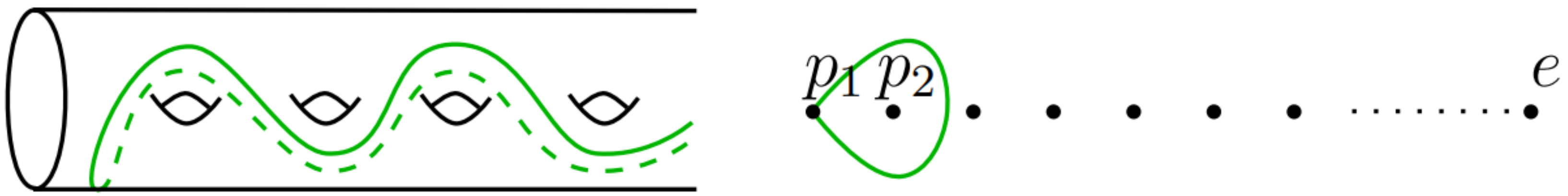

Image by Fanoni–Ghaswala–McLeay

An arc joining distinct ends is omnipresent if it intersects every one-cut homeomorphic subsurface.

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Arc Graph, $\mathcal{A}(\Sigma)$ Vertices: Isotopy classes of essential arcs

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Theorem (Fanoni–Ghaswala–McLeay): For any stable surface Σ with at least three finite-orbit ends, the omnipresent arc graph is a connected δ-hyperbolic graph on which $\mathrm{MCG}(\Sigma)$ acts with unbounded orbits

Mann–Rafi (2019): There exists equivalence classes of ends.

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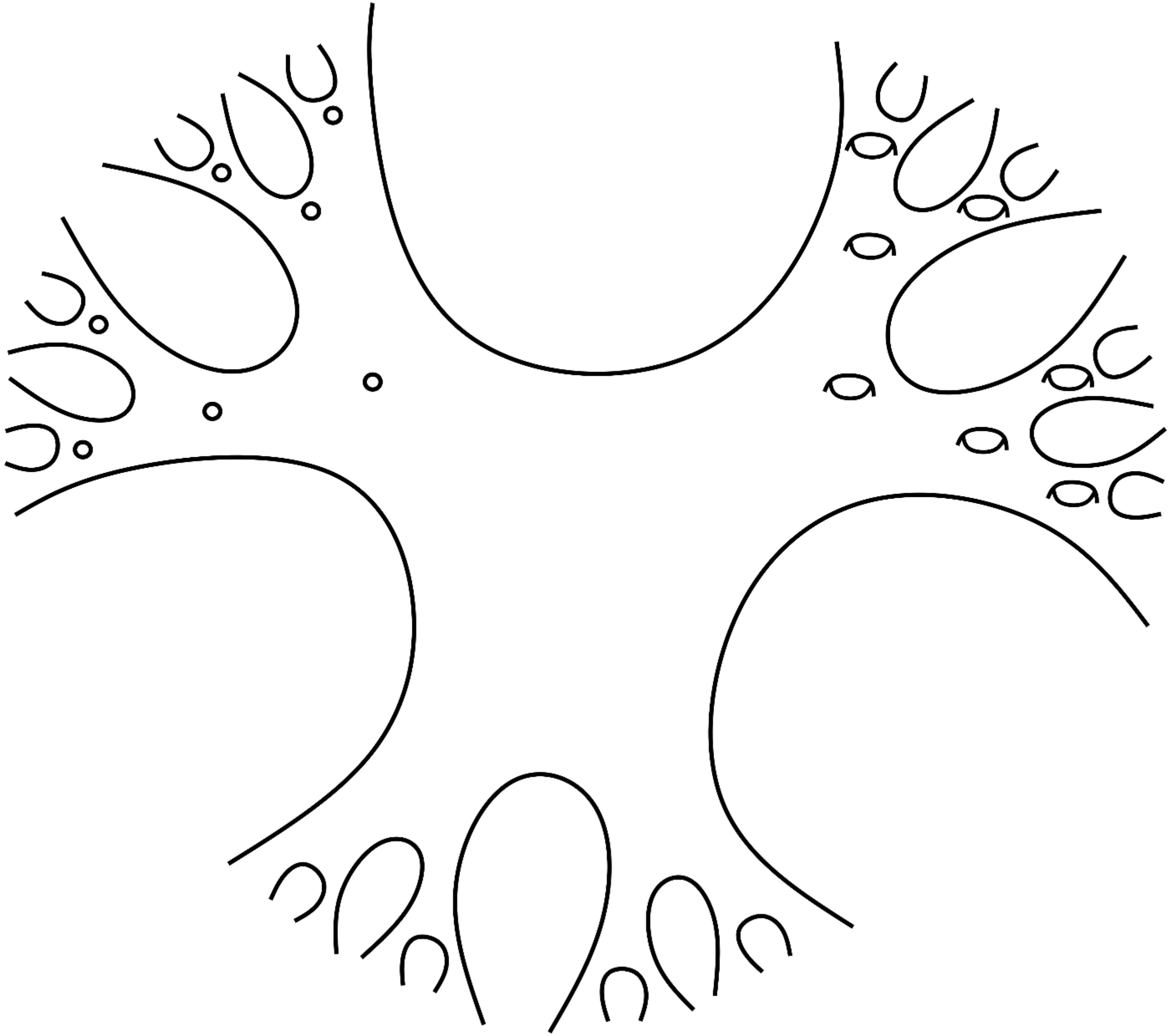
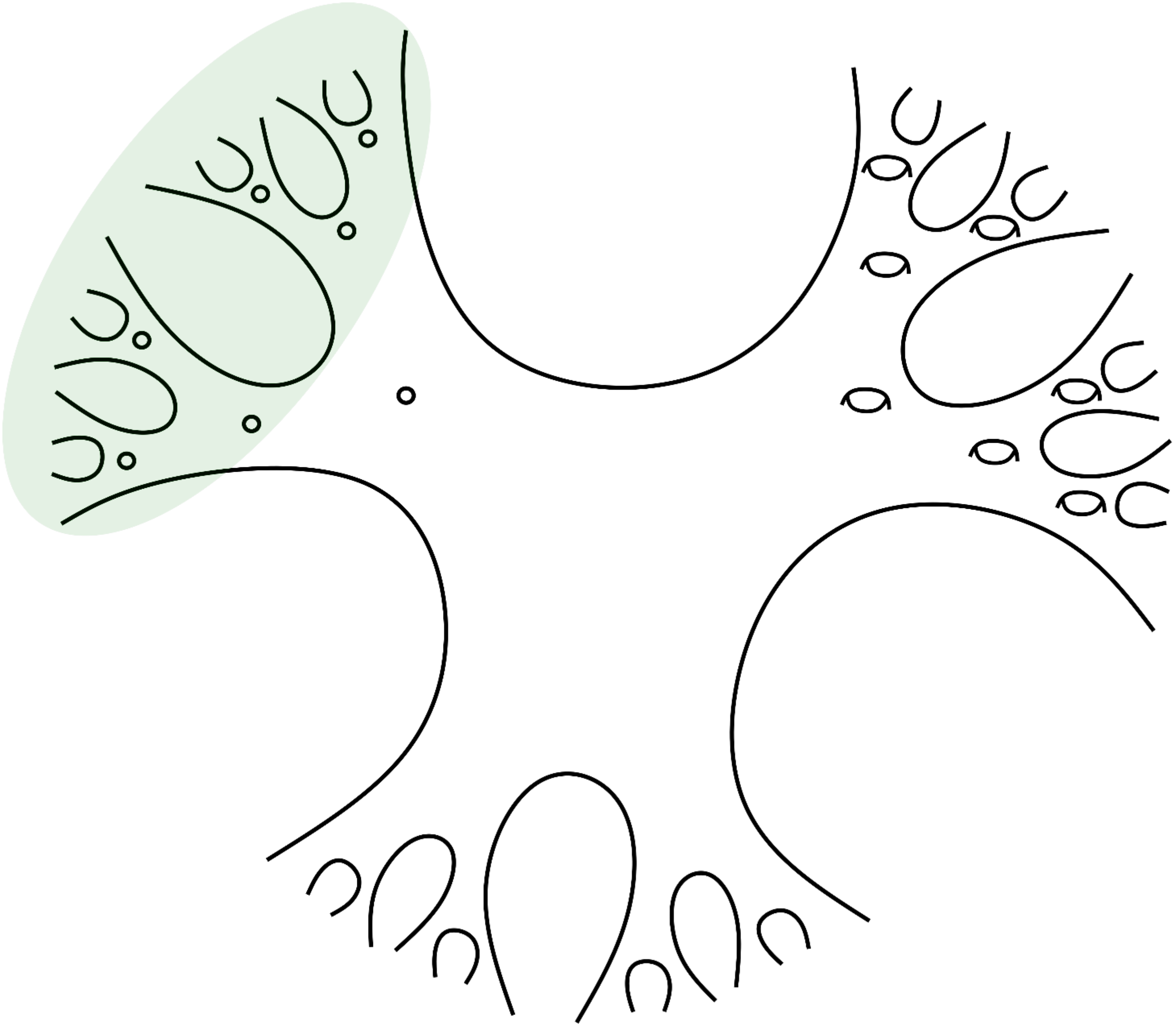
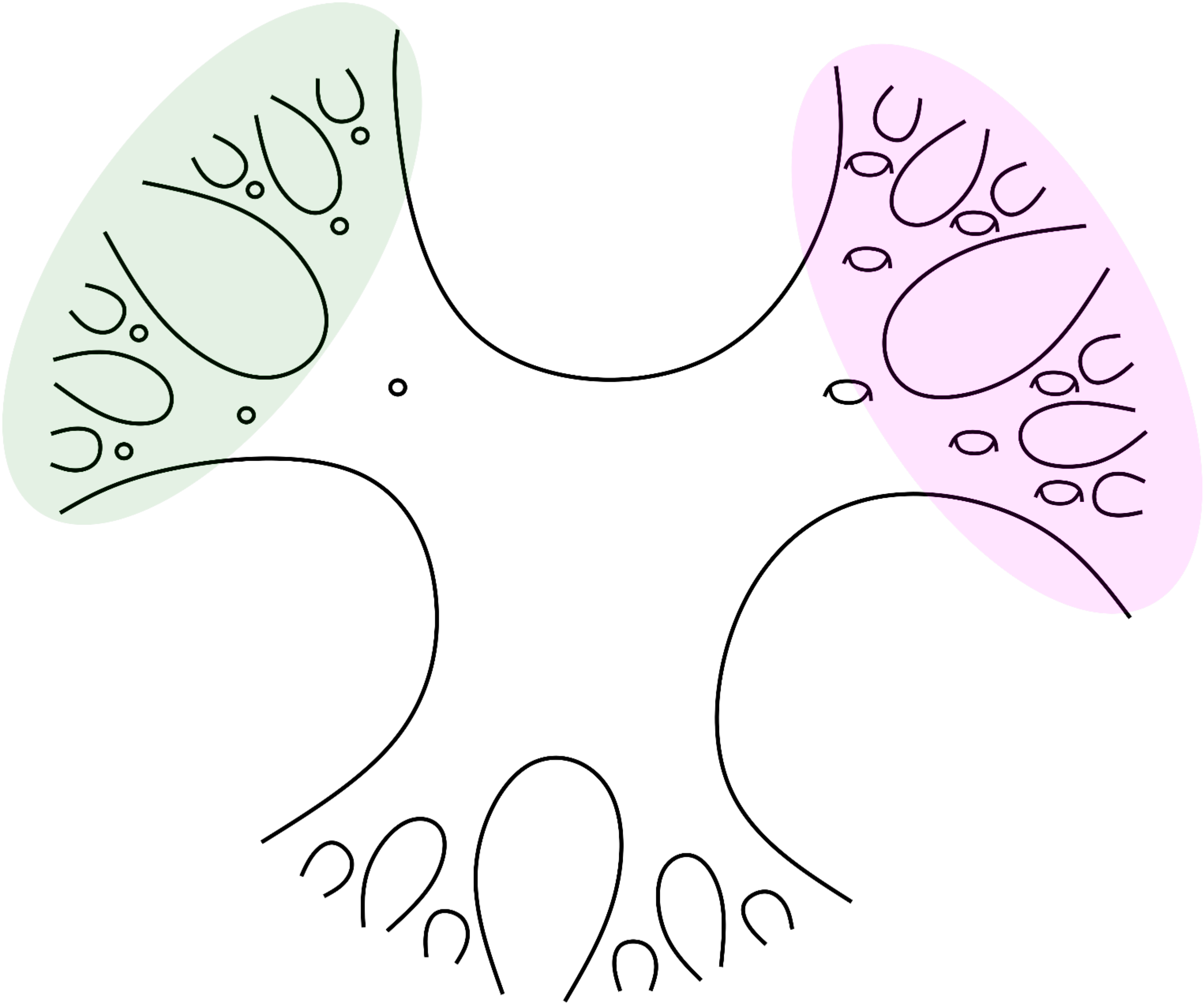
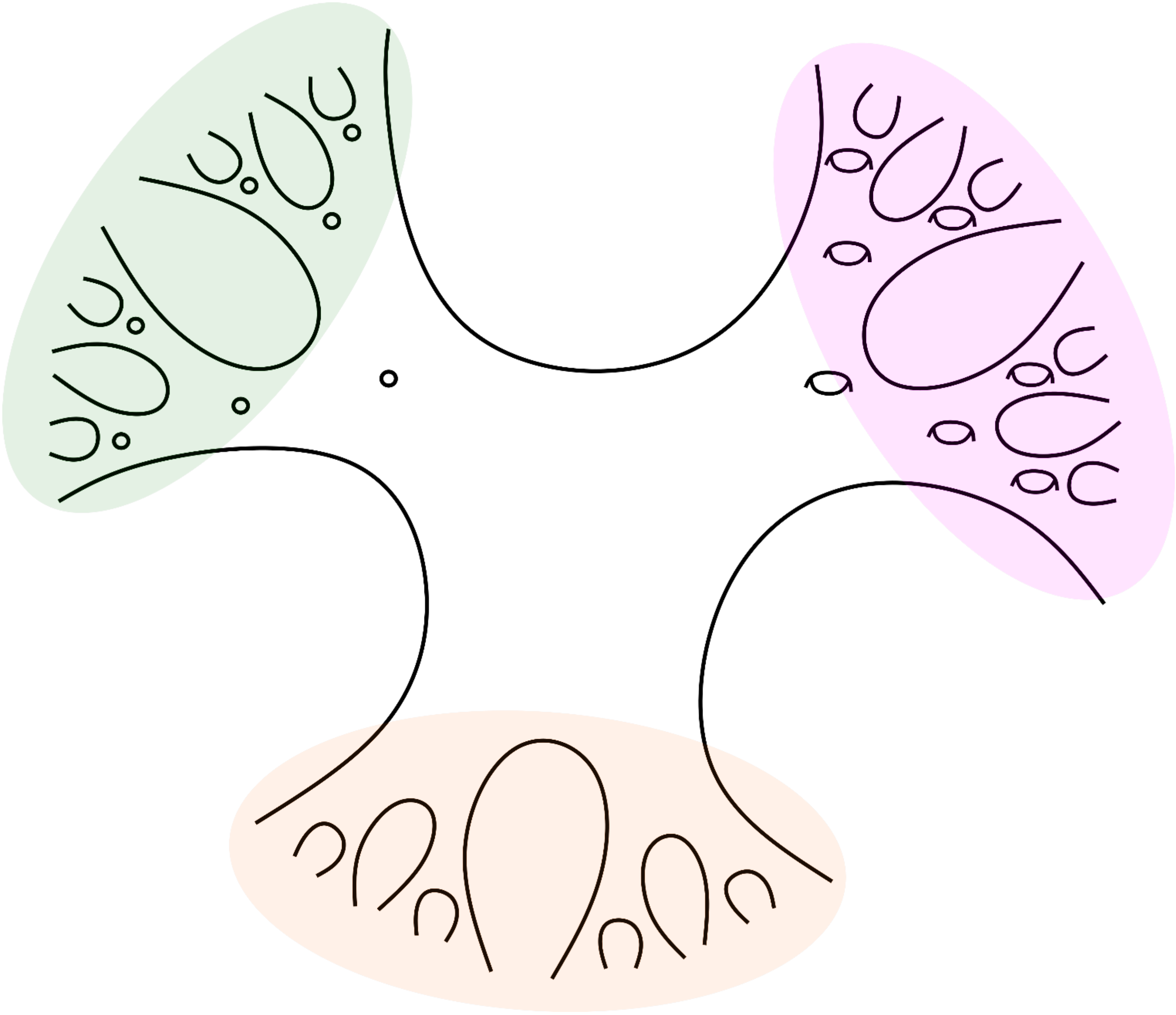
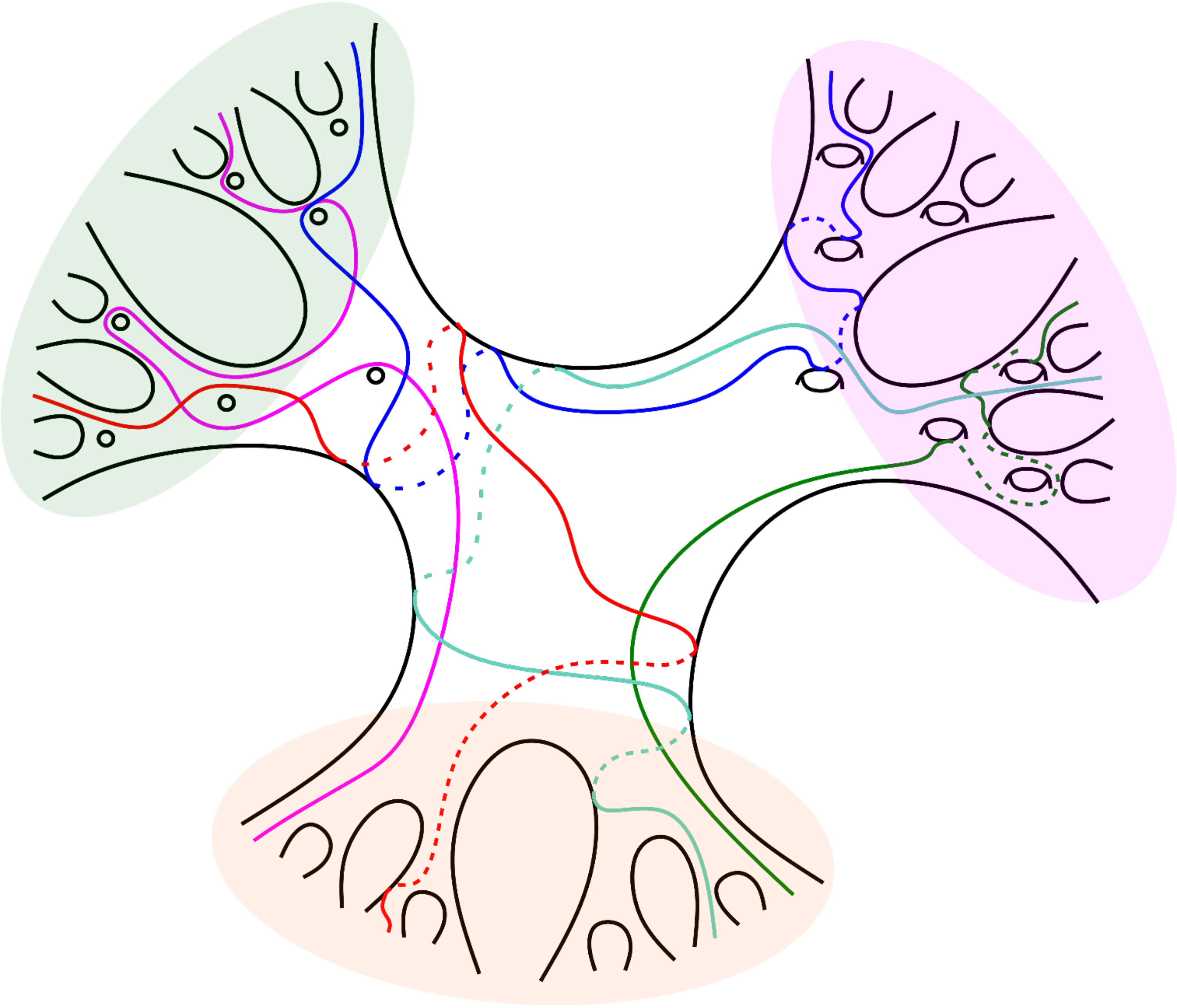
Use these to define a graph using bi-infinite arcs.

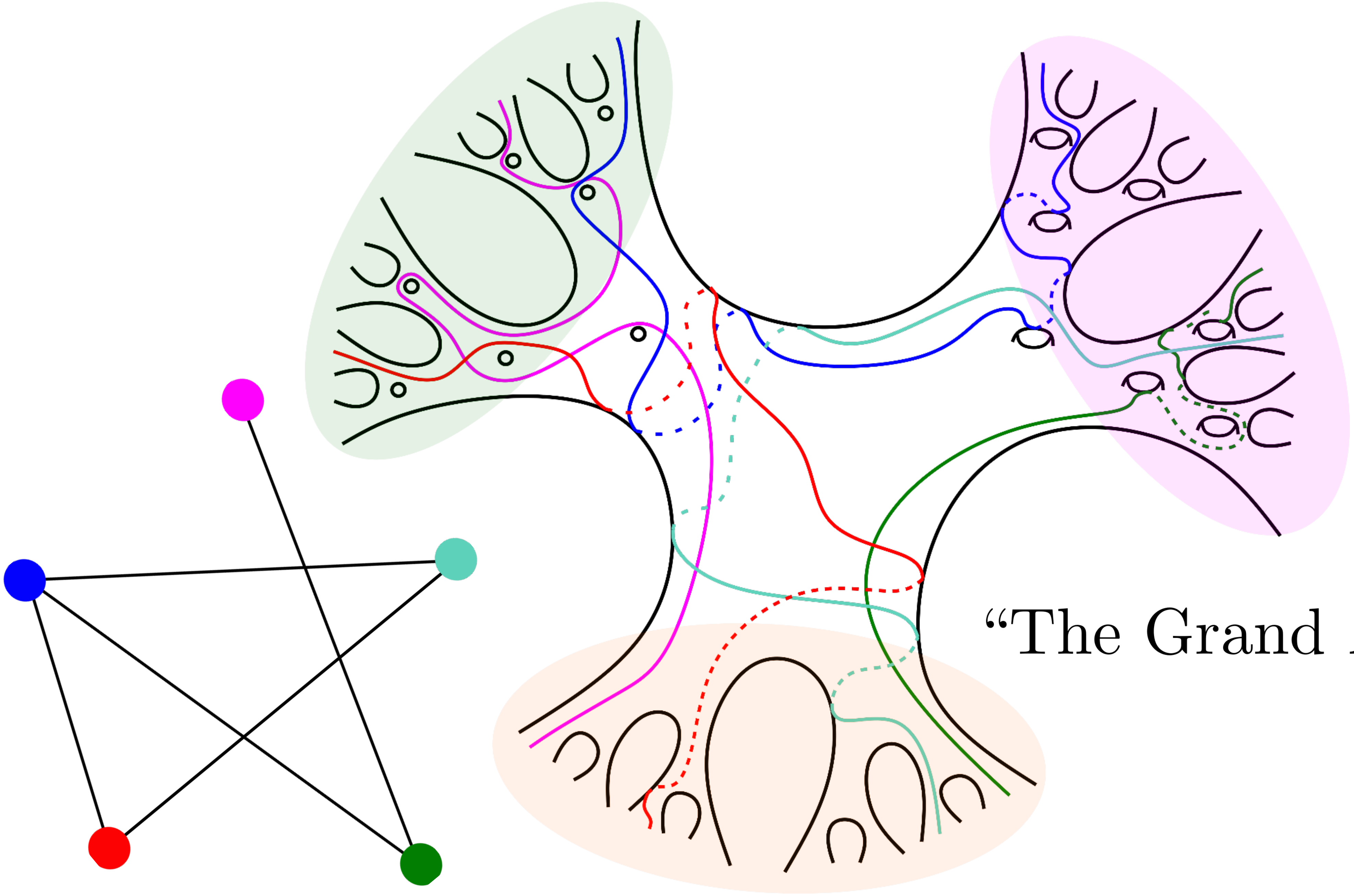










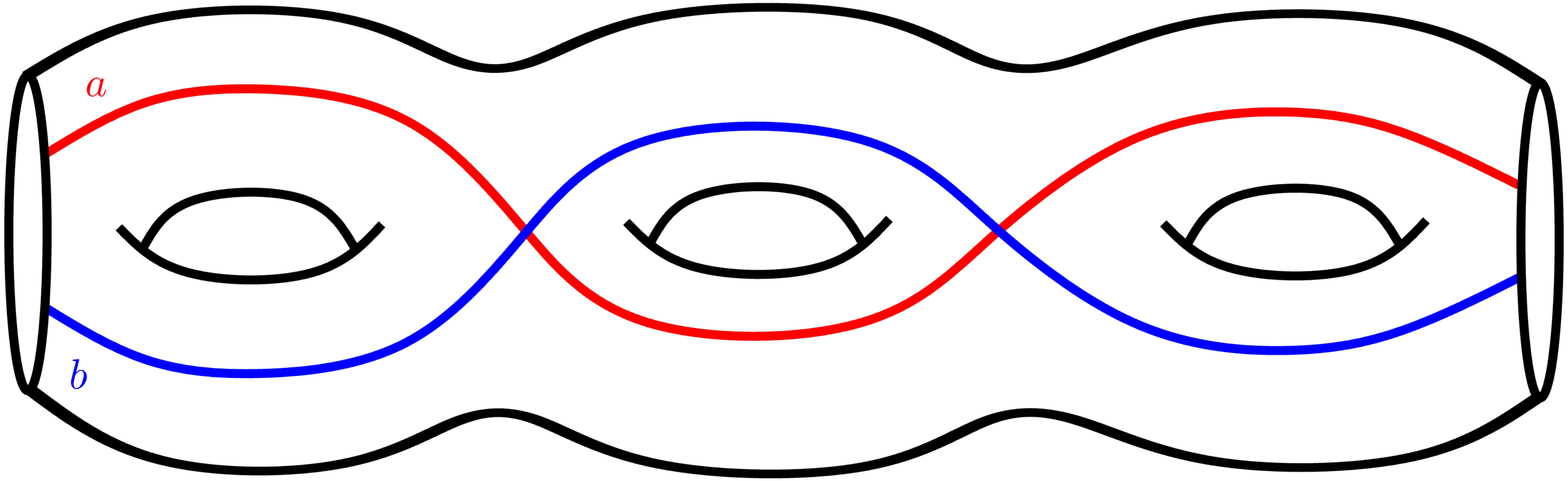


“The Grand Arc Graph”

Theorem (Bar-Natan – V.): For a large class of surfaces, the grand arc graph is connected, hyperbolic, has infinite diameter, and there exist elements of $\mathrm{MCG}(\Sigma)$ which act hyperbolically on the grand arc graph.

Proof Sketch

Unicorn paths (Hensel–Przytycki–Webb [2013]):



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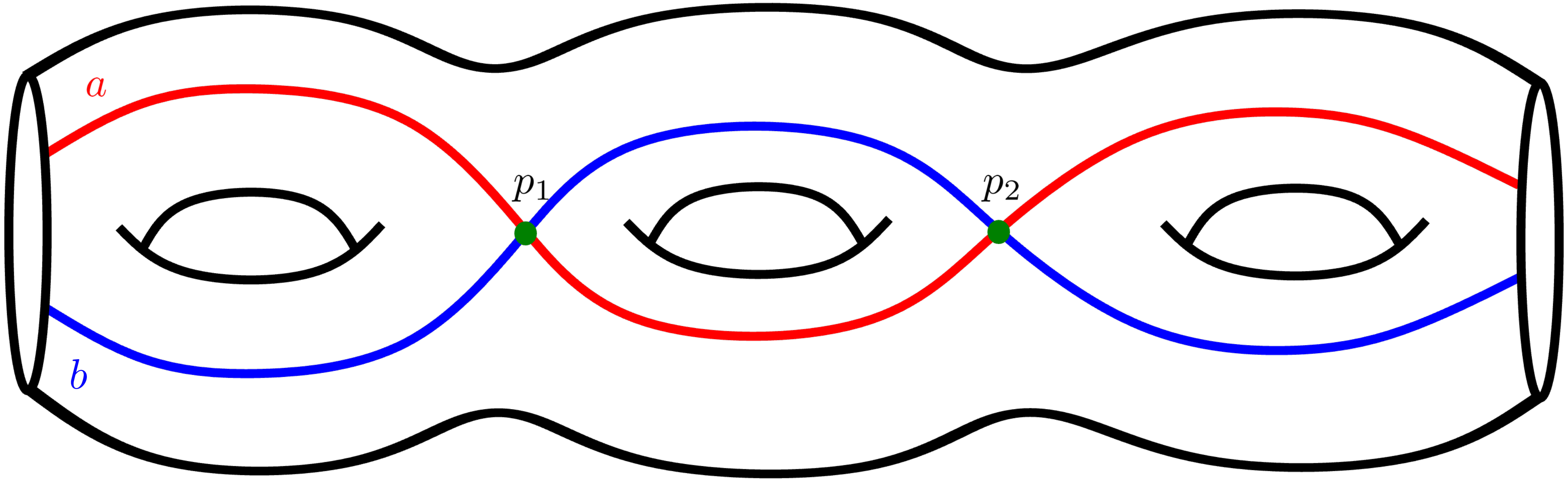

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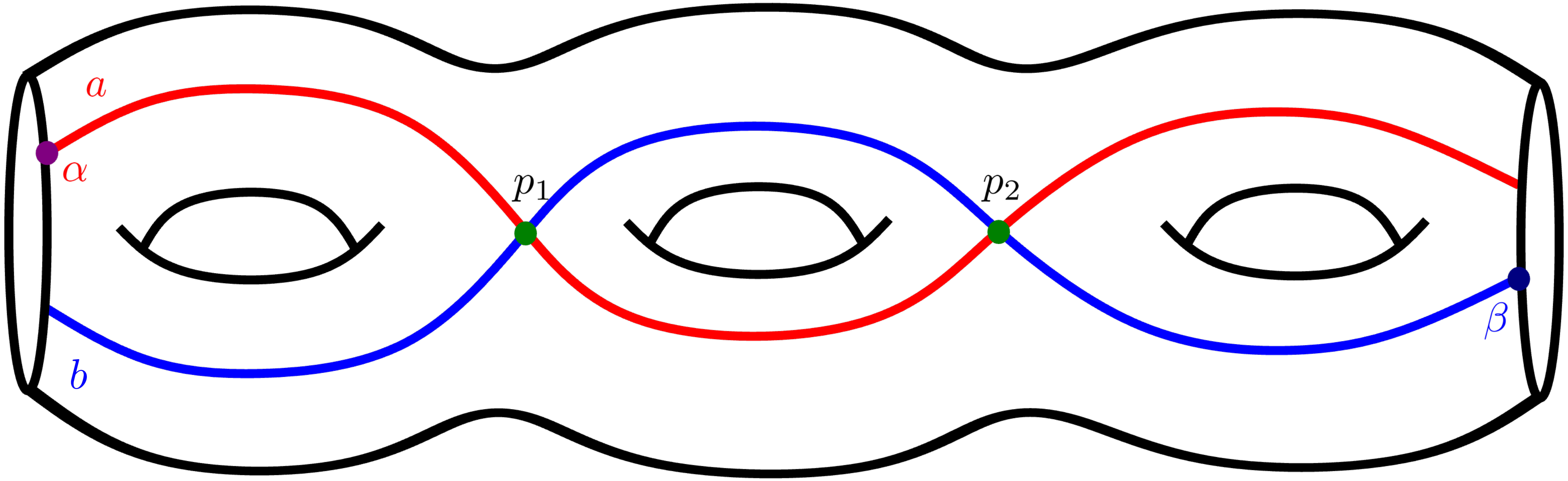

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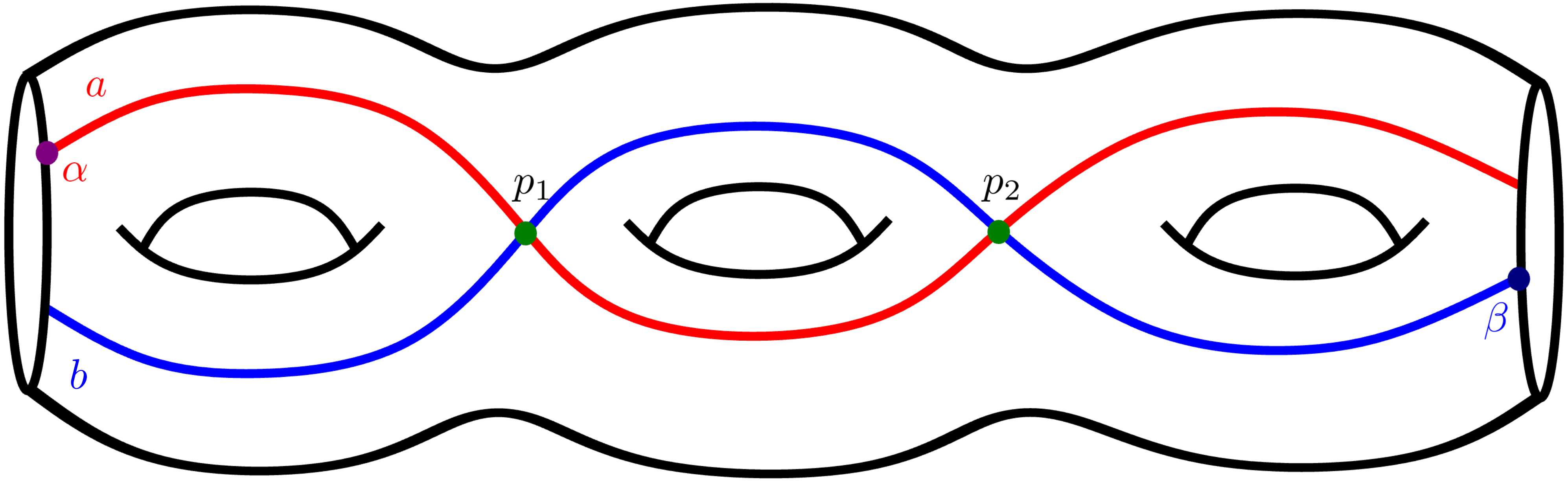
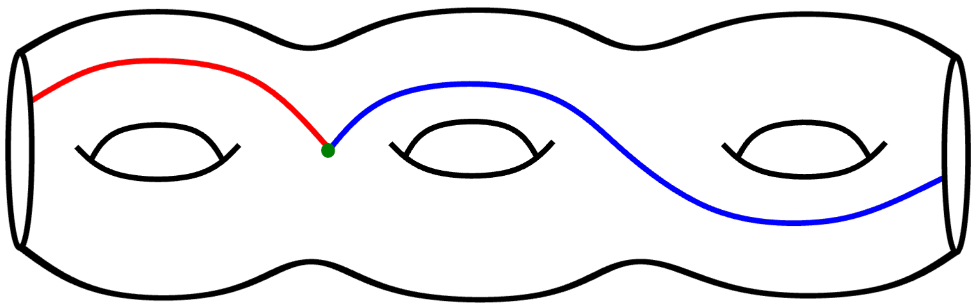


First unicorn arc:



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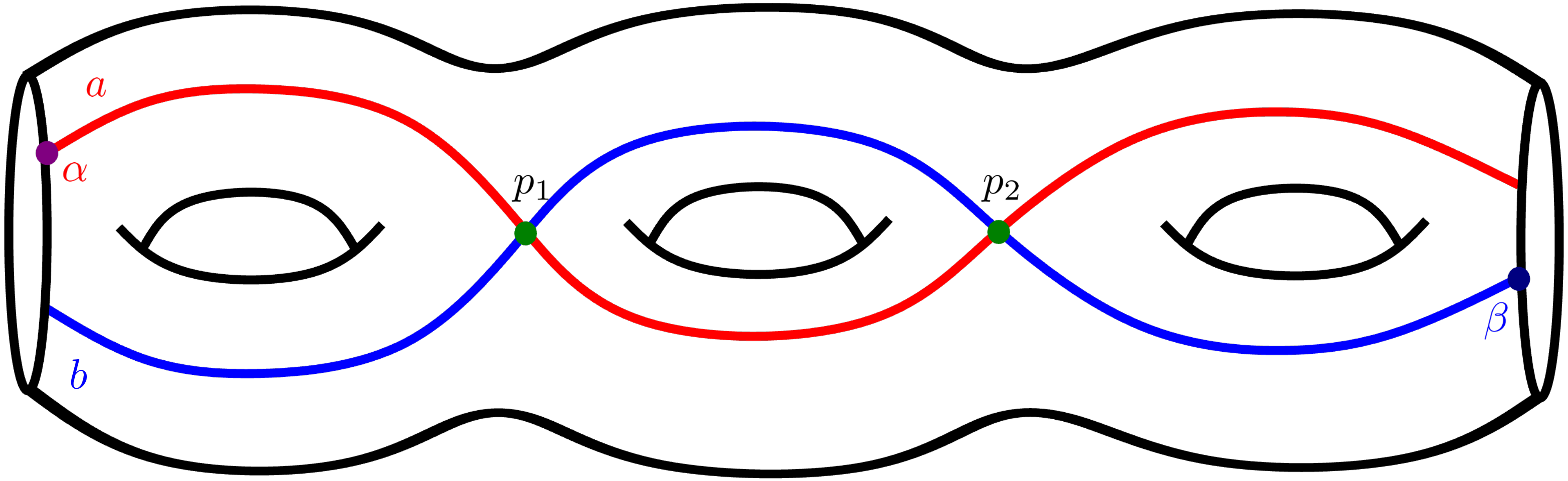

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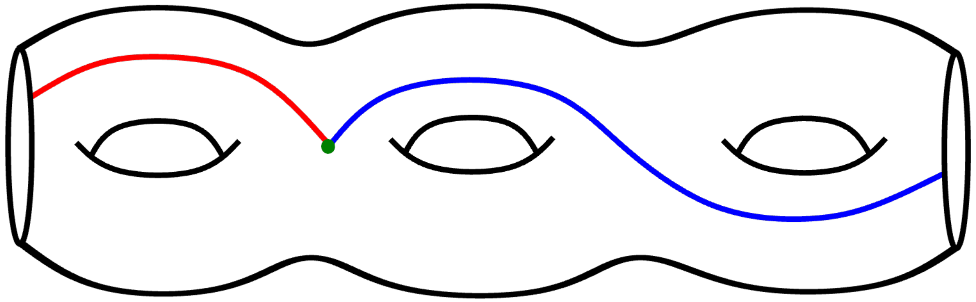
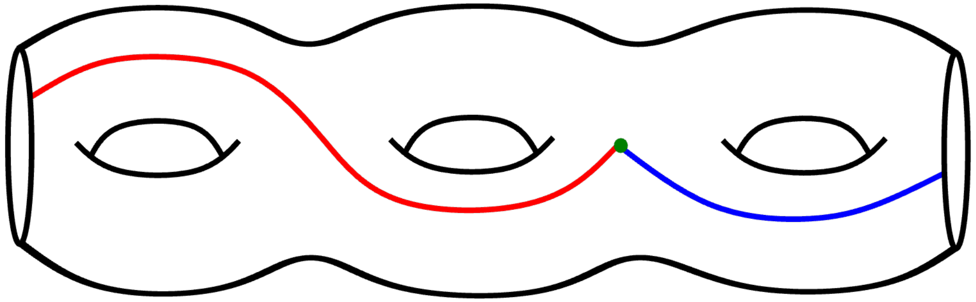


Second unicorn arc:



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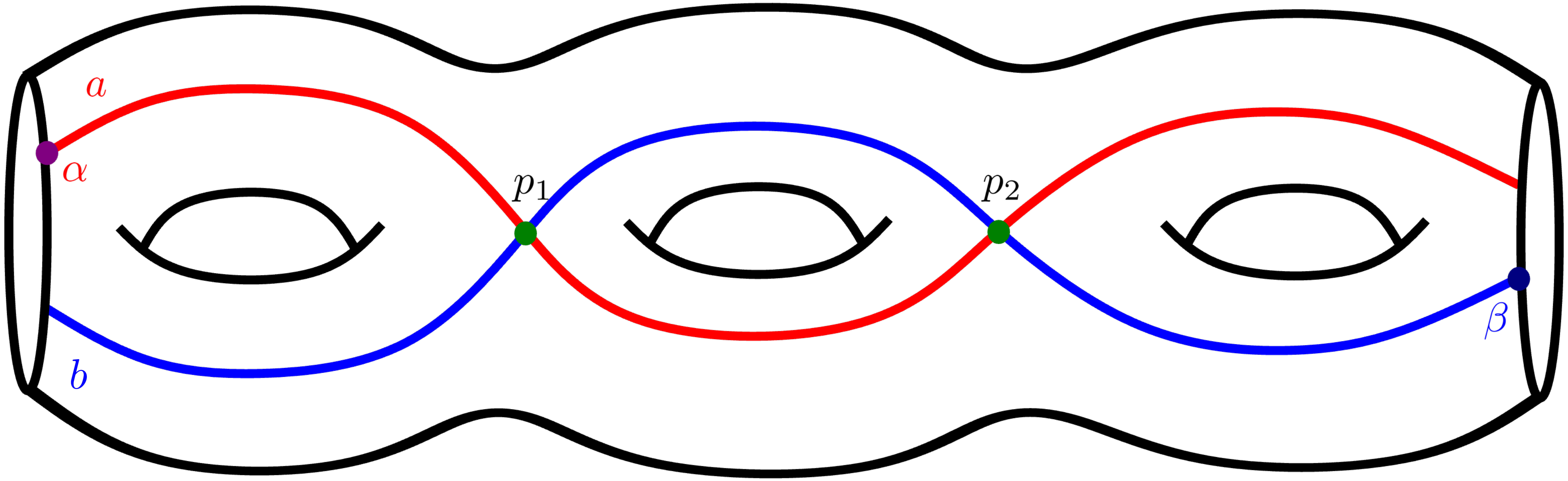

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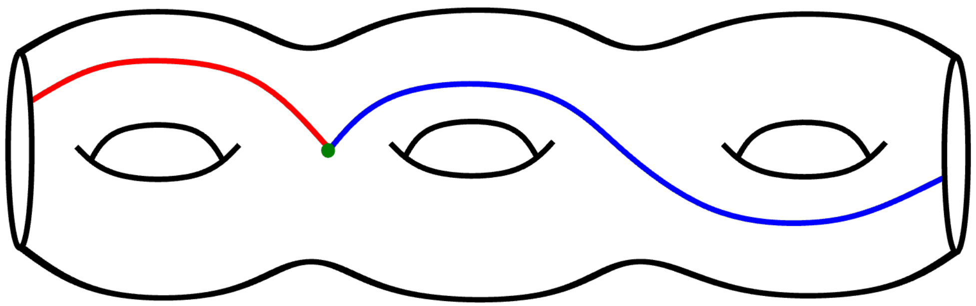

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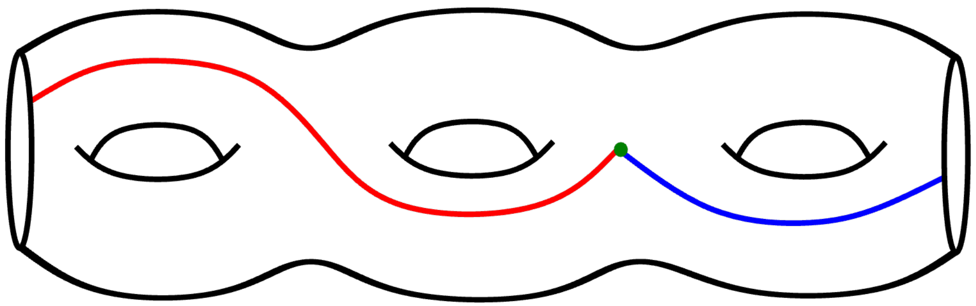

Consecutive unicorn arcs are disjoint.

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Unicorn paths allow us to show the graph is:

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- Hyperbolic

Hyperbolic Actions

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Theorem (Bar-Natan – V.): Let φ be a pseudo-Anosov mapping class that fixes the boundary of W. Let $\bar{\varphi} \in \mathrm{MCG}(\Sigma)$ be the homeomorphism fixing W^c and acting as φ on W. Then $\bar{\varphi}$ acts hyperbolically on $\mathcal{G}(\Sigma)$.